Topics in the Philosophy and Foundations of Mathematics

Lecture 7: The Issue of Realism

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Objects and Truth-Values

Roughly, *realism* is the idea that the *objects/properties* described by mathematical theories 'exist' (so mathematical statements do have a *meaning*).

One often makes a distinction between two main forms of mathematical realism:

- *Object realism (OR)*: mathematical objects exist
- *Truth-value realism (TVR)*: mathematical statements have 'objective' truth-values (are 'objectively true' or 'objectively false')

Of course, one may endorse both OR and TVR.

Object Realism is also commonly referred to as platonism in the literature.

Platonism

[Linnebo, 2018] describes Object Realism (platonism) as the conjunction of the following views:

- **Existence**. There *exist* mathematical objects.
- **Abstractness**. Mathematical objects are not *physically* instantiated (are *non-spatiotemporal*).
- **Independence**. Objects are *independent from* intelligent agents.

There are versions of platonism which weaken A, or I (while holding tight to E): Cantor's platonism, e.g., holds ¬A, whereas Gödel's platonism seems to hold ¬I.

Frege's Argument

Frege's conception of mathematics (cf. [Frege, 1884]), as we have seen (cf. Lecture 1) holds that numbers are 'second-level concepts (number ascriptions)'.

In fact, Frege also thought they were a specific kind of objects (and mathematicians were 'discoverers of such objects', cf. [Frege, 1903]).

Frege's argument runs as follows:

- Classical semantics of (logicised) mathematics yields an 'interpretation' also of *singular terms* within sentences
- For such an interpretation to be 'workable', it is necessary that those terms *refer to* objects
- Since sentences (with terms) may be *true*, then classical semantics implies that *there are* objects
- Conclusion: E

Frege's Realism, Again.

It should be noted, though, that, on Frege's own account of mathematics, E doesn't really explain the *objectivity* of mathematics.

In fact, Frege holds the view: 'mathematical statements are objective, *hence* there exist mathematical objects?.

This accounts for TVR's being seen as a legitimate alternative (or co-existing) view.[1]

TVR has us hold the *objectivity* of maths, without commtting us to E.

[1]TVR seems to have been mentioned, in the first place, by Georg Kreisel, cf. [Dummett, 1978].

Enter Gödel

Gödel's OR is a form of set-theoretic OR (SOR): 'abstract sets exist'.

The fundamental feature of Gödel's SOR is the idea that the construction of V (fostered by the iteration of the 'set of' operation) is guided by a strong *intuition* of the features of V.

Such an intuition, analogous, but inequivalent to, *sensory intuition* is specifically directed at sets, and also globally accounts for STVR (the existence of an objective truth-value for all set-theoretic statements)

Gödel's platonism has been taken to give up (or severely limit the strength of) I (cf., on this, [Parsons, 1990], and [Martin, 2005]), but certainly to subscribe to A, E and TVR.

Gödel's Argument for platonism in [Gödel, 1944]

In [Gödel, 1944], Gödel sketches an argument against Russell's *no-class theory* (and for platonism):

- Russell's identification of the Vicious Circle Principle (VCP), and of *impredicativity*, as the sources of the paradoxes is misleading
- In particular, there is no danger of incurring in the paradoxes by impredicatively iterating the construction of sets (this is, in essence, the 'iterative concept of set')
- While impredicative set theory is very successful at incorporating (reconstructing) the*whole* of mathematics, predicative set theory isn't
- Platonism is hinged on impredicative set theory
- So, Platonism is correct and immune from paradoxes

Gödel's Programme, Again

In [Gödel, 1947], Gödel launches the programme for the search for new axioms (extending ZFC).

One fundamental reason is Gödel's belief in (at least) STVR: he claims that CH has a *determinate truth-value since* CH since ZFC describes a *well-determined reality*.

As we'd seen, Gödel's platonism also entails Universism (there is just one universe of sets), and denies that there are *absolutely undecidable statements*.

The latter claim depends on Gödel's view that the 'incompleteness phenomenon' shows that mathematics is inexhaustible.

Gödel's 'Disjunctive Argument'

In [Gödel, 1951], Gödel also introduces his famous 'disjunctive argument' (purportedly supported by his Incompleteness Theorems):

1. Either the mind surpasses the power of any (finite) machine or
2. There are absolutely undecidable statements

Both (1.) and (2.) support (some form of) platonism about mathematics, insofar as:

- If (1.) is correct, then the mind is non-mechanistic.
- If (2.) is correct, then 'mathematical truths' is a broader realm than that of 'provable mathematical truths'.

Gödel's Intuition

Alternative auxiliary accounts of Gödel's conception of *intuition* and SOR may be found in the literature, which help buttress Gödel's belief in E (in accordance with A and, perhaps, ¬I):

- [Maddy, 1990]'s account: neurophysiologically accountable perception of *impure* sets (cf. Lecture 5). Problem: perception of *pure* sets
- Parsons' account in [Parsons, 1980]: distinction between concrete and quasi-concrete objects. Problem: role of *types*
- [Føllesdal, 1995]'s (and [Hauser, 2006]'s, cf. also [Tieszen, 2012]) invocation of Husserl's phenomenology: 'acts of perception', such as imaginings, re-model perceptual content. Problem: nature of 'phenomenological' method

The Dilemma

[Benacerraf, 1973] has famously argued that there is a tension between:

- I+A.
- (CTK) The causal theory (account) of knowledge.

As a consequence, the platonist would end up adopting a view (I+A) which isn't warranted by our best and most plausible epistemological conceptions.

The epistemology of maths may (for the sake of simplicity) be divided into two factions: supporters of I+A who deny the value of (CTK), or supporters of (CTK) who deny the value of I+A.

Conservative and Non-Conservative Responses

[Panza and Sereni, 2010] classifies conceptions of mathematical knowledge according to their responses to the dilemma as follows:

- Conservative: platonism should be kept, but reformulated so as to make it suitable to respond successfully to the dilemma
- Non-conservative: platonism should be abandoned, and a more workable account of mathematics (eg., one based on CTK) should be adopted.

Conservative responses include:

1. Quine's Holism (and Maddy's Naturalism)
2. Neo-logicism (see Lecture 1)
3. Platonised naturalism
4. Full-blooded (higher-order) Platonism
5. *Ante rem* structuralism

The Quine-Putnam Indispensability Argument

According to Quine, mathematics is part of the broader scientific enterprise (within what he calls the 'web of beliefs').

The whole of science, including *mathematics*, is, thus, confirmed or disconfirmed all together. This is *confirmation holism*.

Applicable maths would, thus, commit us to:

- The objectivity of mathematical statements (TVR)
- E (but different from 'classic' platonist's E)

This is known as the Quine-Putnam *indispensability argument* (cf. [Quine, 1981] and [Putnam, 1971]).

Troubles with Holism and Indispensability

There are problems both with holism and indispensability.

- (*Overkill*) There are mathematical statements which look true regardless of their scientific 'use'.
- (*Inapplicability*) Parts of mathematics are left unjustified (unaccounted) *qua* scientifically dispensable.
- (*Confirmation*) Mathematical statements aren't confirmed like other scientific statements.
- (*Methodology*) There's a mismatch between the methodology of the mathematician and the very idea of confirmation.
- (*Nominalism*) It isn't true that scientific statements quantify over abstract mathematical objects (science can be *nominalised*).[2]

[2] For nominalism in mathematics and science, see [Field, 1980].

Maddy's Set-Theoretic Naturalism

In a fortunate series of books and articles, Maddy has expounded her 'set-theoretic naturalism' (cf. [Maddy, 1996], [Maddy, 1997], [Maddy, 2011]).

By drawing both on Gödel's and Quine's platonism, the position fosters the following views:

- 'Philosophy-first' accounts of and motivations for mathematics ought not to be accepted (a prominently Quinean position)
- Mathematical (set-theoretic) work should be judged in light of *intra-mathematical* (*set-theoretic*) criteria
- Quine's *indispensability argument* is rejected (in fact, reformulated to account for the progress of the discipline)
- Thin (as opposed to Robust) Realism

Extrinsicness Vindicated

- Recall Gödel's distinction between intrinsic and extrinsic arguments ([Gödel, 1947]): Maddy views only *extrinsic* arguments (based on 'fruitfulness', 'simplification', 'depth', etc.) as acceptable
- Intrinsic arguments (relating to the 'concept of set') may just be 'heuristically useful'
- *Intra-mathematical* (set-theoretic) criteria which are relevant to evaluating the progress of the discipline arise from the discipline itself (examples: 'maximise', 'unify', 'meta-mathematical corral', etc.)

Plenitudinous Versions of Platonism

These posit the existence of *many* platonic objects, according with the:

Principle of Plenitude (PP)

There are *as many* mathematical objects *as* can be *consistently* conceived of.

PP takes us well beyond the 'classic' platonic realm.

The trick is to argue that, since there is such a *plenum* of conceivable objects, Benacerraf's argument does not go through (since knowledge of such objects reduces, in a sense, to knowledge of *purely logical* principles needed to produce them).

Full-Blooded (and Higher-Order) Platonism

[Balaguer, 1995] introduces the following principle:

For any theory Γ*, there exists a class* C *of objects which satisfies* Γ

The principle may be paraphrased as follows: any theory describes a portion of the 'mathematical universe'.

In other terms, if it's possible (= consistent) for an object O to exist, then there is a theory T which describes it. So, all objects which *can* exist *do* exist.

This has been seen as a re-statement of a Hilbertian conception of mathematics (see, e.g., the criticisms of Balaguer's FBP in [Potter, 2004]), since, by FBP, existence = consistency.

It isn't clear, in particular, how FBP also meets the classic platonist's E, A, I postulates.

Hamkins' Higher-Order Platonism

A slightly different FBP-ist approach to set theory is Hamkins' platonic pluralism ([Hamkins, 2012], [Hamkins, 2020]). On Hamkins' view, there exist *second-order (higher-order)* platonic objects, such as universes, collections thereof, etc.

This allows Hamkins to advocate the following views:

- There are *many* concepts of set.
- There is no single, 'preferred', universe of sets, rather a 'multiverse'.
- CH (and any other undecidable statement) is solved.
- Pluralism extends the theory/metatheory relationship to a hierarchy of metatheories.[3]

[3] A theory comes with a metatheory where one explores models which 'realise' other theories, and so on. Cf. [Hamkins, 2020], p. 477ff.

Responses to Hamkins

The multiverse has both logical and epistemological problems.

- 'Multiverse of a theory T' is an ill-defined (at best, incomplete) concept.
- Multiversism implies an infinite definitional regress along the consistency strength hierarchy of theories (cf. [Koellner, 2013]).[4]
- Distinctions among alternative 'concepts of set' are not sharp (and there's just too many useless distinctions)
- Knowledge of such (higher-order) objects seems to be along the lines of FBP's (purely 'conceptual') account, so problematic

[4] To articulate the multiverse of T, one needs assume $Con(T)$, but, in order to do that, one needs assume $(Con(T + Con(T))$ and so on.

Platonised Naturalism

A different version of the plenitude principle is that considered by Zalta and Linsky in their Object Theory ([Zalta, 1983], [Linsky and Zalta, 1995]).

The two introduce a different kind of platonic objects and form of 'predication': *encoding*.

The central axiom is a (logical) Comprehension Principle:

$$(\exists x)(\mathcal{A}(x) \wedge Abs(x) \wedge (\forall P)[(x)P \leftrightarrow \mathcal{A}]$$

where $x(P)$ means: 'x encodes P', $Abs(x)$ ='x is abstract', and $\mathcal{A}(x)$ is any formula.

So, for each property or properties P there is exactly an abstract object which 'encodes' them.

The mathematical objects are those which satisfy the axioms of Linsky-Zalta's Object Theory.

Platonised Naturalism, Again

Ultimately, Linsky and Zalta restrict the range of mathematical objects to those which encode 'properties which are true of that object by a theory T' (naturalism).

$$\kappa_T =_{df} \iota x(Abs(x) \wedge (\forall P)((x)P \leftrightarrow T \models P(\kappa_T)))$$

So, the 2_{PA} is the unique number which has the properties of '2' (by PA).

Problems:

- Existence of (mathematical) objects is contingent on existence of *theories* of those objects (objects are *places* in structures).
- Description (definition) of theories doesn't seem to meet naturalistic standards.
- Again, it isn't clear that Benacerraf's dilemma is overcome.

Ante Rem Structuralism

Structuralism is the view that mathematicians deal with structures, not with objects.

Ante rem structuralism (cf. [Shapiro, 1997]) is the view that such structures satisfy the platonist's E, A, and I.

That is: the 'structure of natural numbers', that of 'real numbers', of 'sets', etc. are independently existing abstract mathematical structures.

The position, among other things, helps respond to Benacerraf's 'other' problem about different representations of the same numbers (cf. [Benacerraf, 1965]).

ARS, again

ARS is a response to Benacerraf's dilemma in the sense that the 'ability to think of a structure' would be evidence that such structures exist.

This is, for instance, [Shapiro, 1997]'s argument (Chapter 4, but also cf. [Parsons, 1990]).

Other *non-eliminative structuralists*[5] think that the process of 'pattern recognition' lying at the roots of 'knowledge of structures' could be causally (empirically) accounted for.

This is, for instance, [Resnik, 1997]'s view. But the latter seems to adopt, and be affected by the same problems as, Quinean (or Maddy's first version of) realism.

[5]Non-eliminative structuralism is the view that structures *exist* (as opposed to 'eliminative' structuralism which posits that structures are just ways to represent specific bits of mathematical thinking).

This Lecture's Main Sources

- [Linnebo, 2017]
- [Linnebo, 2018]
- [Shapiro, 2005]
- [Hamkins, 2020]
- [Panza and Sereni, 2010]

End

Thanks for attending!

Balaguer, M. (1995).
A Platonist Epistemology.
Synthese, 103:303–25.

Benacerraf, P. (1965).
What Numbers Could Not Be.
The Philosophical Review, 74(1):47–73.

Benacerraf, P. (1973).
Mathematical Truth.
The Journal of Philosophy, 70(19):661–679.

Dummett, M. (1978).
Truth and Other Enigmas.
Harvard University Press, Cambridge (MA).

Field, H. (1980).
Science Without Numbers. A Defense of Nominalism.
Princeton University Press, Princeton (NJ).

Føllesdal, D. (1995).
Gödel and Husserl.
In Hintikka, J., editor, *From Dedekind to Gödel: Essays on the Development of the Foundations of Mathematics*. Kluwer, Dordrecht.

Frege, G. (1884).
The Foundations of Arithmetic.
Pearson-Longman, New York.

Frege, G. (1893-1903).
Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet, volume I-II.
Pohle, Jena.

Gödel, K. (1944).
Russell's mathematical logic.

In Schilpp, A., editor, *The Philosophy of Bertrand Russell*, chapter Russell's Mathematical Logic, pages 125–153. Open Court Publishing, Chicago.

Gödel, K. (1947).
What is Cantor's Continuum Problem?
American Mathematical Monthly, 54:515–525.

Gödel, K. (1951).
Some Basic Theorems on the Foundations of Mathematics and its Applications.
Gibbs Lecture.

Hamkins, J. D. (2012).
The Set-Theoretic Multiverse.
Review of Symbolic Logic, 5(3):416–449.

Hamkins, J. D. (2020).
Lectures on the Philosophy of Mathematics.
MIT Press, Cambridge (MA).

Hauser, K. (2006).
Gödel's Program Revisited. Part I: The Turn to Phenomenology.
The Bulletin of Symbolic Logic, 12(4):529–90.

Koellner, P. (2013).
Hamkins on the Multiverse.
Pre-print.

Linnebo, Ø. (2017).
Philosophy of Mathematics.
Princeton University Press, Princeton.

Linnebo, Ø. (2018).
Platonism in the Philosophy of Mathematics.

Linsky, B. and Zalta, E. (1995).
Naturalized Platonism versus Platonized Naturalism.
Journal of Philosophy, XCII(10):525–555.

Maddy, P. (1990).
Realism in Mathematics.
Clarendon Press, Oxford.

Maddy, P. (1996).
Set-Theoretic Naturalism.
Bulletin of Symbolic Logic, 61(2):490–514.

Maddy, P. (1997).
Naturalism in Mathematics.
Oxford University Press, Oxford.

Maddy, P. (2011).
Defending the Axioms.
Oxford University Press, Oxford.

Martin, D. (2005).
Gödel's Conceptual Realism.
Bulletin of Symbolic Logic, 11(2):207–224.

- Panza, M. and Sereni, A. (2010).
Il problema di Platone. Un'introduzione storica alla filosofia della matematica.
Carocci, Roma.

- Parsons, C. (1980).
Mathematical Intuition.
Proceedings of the Aristotelian Society, 80:145–68.

- Parsons, C. (1990).
The structuralist view of mathematical objects.
Synthese, 84(3):303–346.

- Potter, M. (2004).
Set Theory and its Philosophy.
Oxford University Press, Oxford.

- Putnam, H. (1971).
Philosophy of Logic.
Harper, New York.

Quine, W. V. (1981).
Theories and Things.
MIT Press, Cambridge (MA).

Resnik, M. (1997).
Mathematics as a Science of Patterns.
Oxford University Press, Oxford.

Shapiro, S. (1997).
Philosophy of Mathematics. Structure and Ontology.
Oxford University Press, Oxford.

Shapiro, S., editor (2005).
Oxford Handbook of Philosophy of Mathematics.
Oxford University Press, Oxford.

Tieszen, R. (2012).
Monads and Mathematics: Gödel and Husserl.
Axiomathes, 22:31–52.

Zalta, E. (1983).
Abstract Objects: an Introduction to Axiomatic Metaphysics.
Reidel, Dordrecht.