Topics in the Philosophy and Foundations of Mathematics

Lecture 6: Alternative Conceptions of the Infinite

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Conceptions of the Infinite

Historically, different conceptions (philosophies) of the infinite have been articulated (cf., e.g., [Mancosu, 2009]).

Set theory may be seen as the final affirmation of the *actual infinite* as based on Cantor's Principle.

However, even within the very realm of the actual-infinite-based mathematics, other (sometimes, richer) ways of dealing with, for instance, 'infinite measures' or 'cardinalities' are possible.

Some of these openly aim to rescue 'infinitesimals' which had been eschewed from mathematics by the XIX century analysts and considered irretrievably mistaken by Cantor himself.

Why ‘alternative’ conceptions?

What may justify the adoption of alternative conceptions?

- Anti-platonism (underlying assumption: set theory expresses the platonistic point of view in mathematics, cf. [Dummett, 1977] and [Feferman, 1987])
- Dissatisfaction with set-theoretic conception of the *continuum*
- Dissatisfaction with set-theoretic undecidability
- Preference for Part-Whole Principle over ‘bijection method’
- Need to allow for the incorporation of *non-archimedean* quantities (ie., *infinitesimals*)
- ...

Brouwer's Potentialism

- Constructivists (in particular, *intuitionists*) believe that only the *potential* infinite is acceptable.
- Brouwer accepts Cauchy sequences,[1] but only to the extent that these are given by *rules*
- Brouwer's 'free-choice' account of the reals envisages the 'creation' of real numbers through 'freely picking' decimal digits ([Brouwer, 1913], [Dummett, 1977])
- Brouwer takes $\aleph_1$ or $\mathbb{R}$ (and, for that matter, any uncountable cardinal $> \aleph_0$) to just be 'denumerably unfinished' collections ($=$ their elements cannot be enumerated)

[1] A Cauchy sequence is a *convergent* sequence of real numbers, that is, a sequence of reals $S = \{a_n\}$ such that, for all $\epsilon > 0$, there exists M such that for all $m, n > M$, $|a_n - a_m| < \epsilon$.

[Weyl, 1918]'s Conception

- The 'whole of' $\mathbb{N}$ is admissible
- Only *predicatively definable* subsets of $\mathbb{N}$ are admissible[2]
- Weyl's system is equivalent to the second-order arithmetic system ACA_0 (cf. [Simpson, 2009])
- [Weyl, 1918] and subsequent developments (such as those in [Feferman, 1987]) aim to show that 'scientifically applicable mathematics' may be developed in some extension of Weyl's predicative system of analysis
- On such grounds, [Feferman, 1987] concludes that Cantor might 'not be necessary'

[2] These are sets definable with an arithmetical formula which quantifies over *all* natural numbers, but not over *all* subsets of natural numbers.

Galileo's Skepticism

- Recall Galileo's Paradox ('there are as many natural as square numbers')
- Galileo expressed skepticism about the possibility of defining 'relations of order' in the infinite (cf. [Galileo, 1638], *Prima Giornata*)
- Thus, to all practical purposes, there should just be *one* infinite
- Peano (in [Peano, 1891] and [Peano, 1892]) also seems to adopt this view
- Galileo's (and Peano's) conception might be called: 'single-cardinality conception of the infinite'

Mancosu's Alternative Principles: Peano's and Boolos' Principles

[Mancosu, 2016] introduces (and, to some extent, advocates) principles alternative to Cantor's ('$\mathfrak{s}(X)$'='size of X'):

Peano's Principle (PP)

$$\mathfrak{s}(X) = \begin{cases} n, & \text{if } X \text{ has } n \text{ elements} \\ \infty, & \text{if there is no injection from } X \text{ to } n \text{ for all } n \end{cases}$$

Boolos' Principle (BP)

$$\mathfrak{s}(X) = \begin{cases} n, & \text{if } X \text{ has } n \text{ elements} \\ \infty_1, & \text{if } X \text{ is cofinite in } \mathbb{N} \text{ (ie., } \mathbb{N} - X \text{ is finite)} \\ \infty_2, & \text{if } X \text{ is not cofinite in } \mathbb{N} \text{ (ie., } \mathbb{N} - X \text{ is not finite)} \end{cases}$$

Mancosu's Principles Under Scrutiny

The main reason behind the introduction of those principles is to put pressure on the neo-logicists' claim of the *analyticity* of:

Hume's Principle

$$(\forall F)(\forall G)\ \#x\ F(x) = \#x\ G(x) \leftrightarrow F \approx G$$

and, thus, also of Cantor's Principle. Since PP and BP may also be expressed as *second-order abstraction principles*, then, Mancosu argues, they could be seen as genuine, alternative *cardinality principles*.

The force of Mancosu's argument depends on what one takes 'numeric abstraction' and 'numeric abstraction principle' to mean.

Buzaglo's Principle

[Buzaglo, 2002]'s $<$ and $\not<$

$|A| < |B|$ iff: (i) there is an injection f from A to B; (ii) there is a $b \in B$, such that $b \notin ran(f)$; (iii) there is no function $g > f$, that is, such that $ran(f) \subset ran(g)$ and $b \in ran(g)$. If, at least, one of (i-iii) is not satisfied, then $|A| \not< |B|$.

Now Buzaglo defines:

Buzaglo's Principle (BuP)

$|A| = |B| \leftrightarrow |A| \not< |B|$ and $|B| \not< |A|$.

Consequences:

- All sets uncountable by CP are countable by BuP.
- So *all* sets are, at most, *countable* by BuP.

Vopěnka's Two-Cardinality Theory

Vopěnka's 'alternative' set theory ([Vopěnka, 1979]) has both sets and classes.

Sets are, at most, *countable*. Classes may be *finite*, *countable* or *uncountable*.

Vopěnka's Axiom of Cardinalities

Any two uncountable classes have the same size.

So, Vopěnka's theory only envisages two cardinalities in the infinite: $\aleph_0$, and $\mathfrak{c}$.

Motivation (cf. [Randall Holmes, 2017]): mathematicians (scientists) fundamentally deal with sets of those sizes.

Infinitesimals Reconsidered

Remember that the:

Eudoxus-Archimedes' Axiom (EAA)

For all reals *a*, *b*, whenever $a > b$, there exists $n \in \mathbb{N}$, such that $nb > a$.

is incompatible with the existence of infinitesimals.

Starting in the second half of the XIX century, several theories have been produced which are able to incorporate infinitesimals:

- Veronese's theory of the linear continuum (subsequently re-worked by Levi-Civita)
- Robinson's non-standard analysis
- Conway's theory of surreals (absolute linear continuum)
- Numerosity theorists' α-theory

Non-Archimedean Fields

A non-archimedean field is a *real-closed field* (a *field* which contains the reals) which contradicts EAA.

Do such fields exist?

Using the Compactness Theorem of first-order logic and an ultrafilter construction, [Robinson, 1966] proved, that if the *first-order* axioms of the field of the reals have a model ($\mathbb{R}$), then they have a (*non-standard*) model with *infinitesimals* and *infinites* ($\mathbb{R}^*$).[3]

The latter are called *hyperreals*.

[3] Compactness Theorem (first-order logic): given a set of sentences T, if every finite subset of T has a model, then T has a model.

$\mathbb{R}^*$

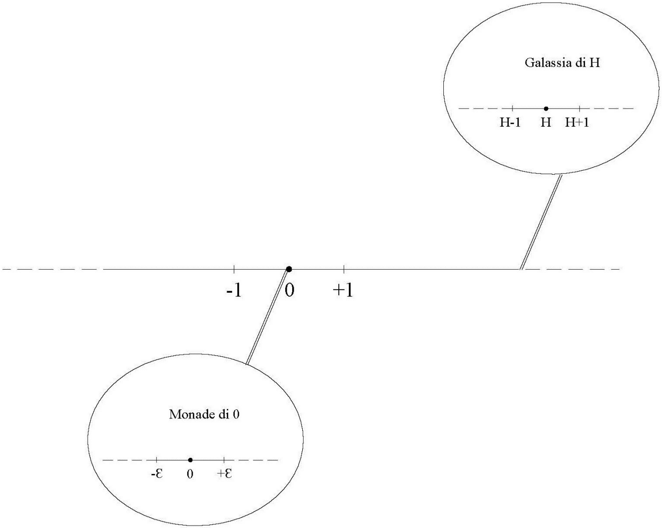
$\mathbb{R}^*$ contains:

- Finite numbers $r, s, t, \ldots$
- Infinitesimals (like ϵ): for all $r \in \mathbb{R}$, $\epsilon < r$.
- Infinite (like H) such that for all $r \in \mathbb{R}$, $H > r$.

These may be combined to produce other numbers:

- $\epsilon + \delta$ and $\epsilon \cdot \delta$ are infinitesimals.
- $H + K$, $H \cdot K$ are infinite.
- $\epsilon + b$ is finite, $\epsilon \cdot b$ is infinitesimal.
- $H + b$, $H \cdot b$ are infinite
- ϵ/δ, H/K are *indeterminate*
- ...

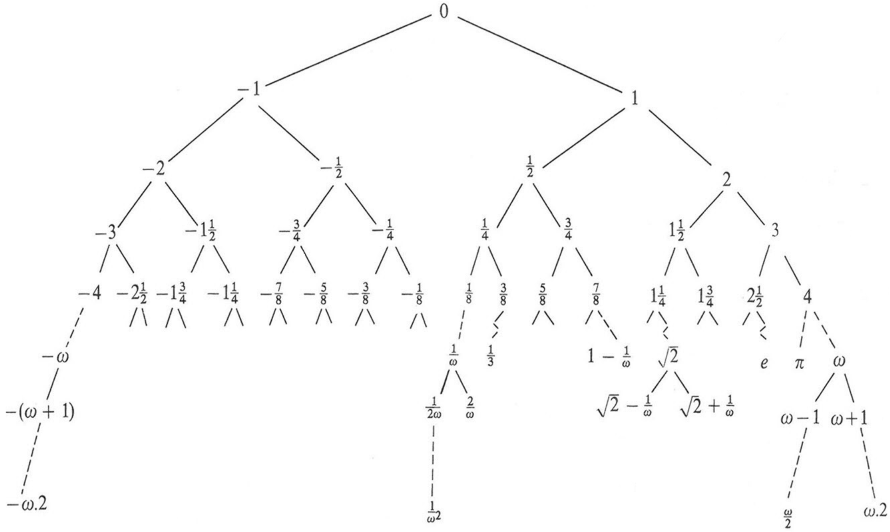
Monads and Galaxies



Surreal Numbers

- In the 1970s, John Conway produced the non-archimedean fields of *surreals* ([Conway, 1976]), denoted **No**.
- This consists of all Dedekind cuts $(A|B)$, where A and B are sets of previously defined numbers, such that, for all $a \in A$, and $b \in B$, $a < b$
- Eg. $\epsilon = (\emptyset|1, 1/2, 1/4, 1/8, ..., 1/2^n, ...)$.
- Each row ('day') of the tree of the surreals (N) contains numbers born on that day (see next slide)
- $N_0 = \{\emptyset\}$, $N_1 = \{-1, 1\}$, ...,
- It can be proved that to each surreal can be assigned a unique *birthday*
- One may even continue to count past Ord: $Ord = (\mathbf{No}|\emptyset)$, $Ord + 1 = (Ord|\emptyset)$, ...

Tree of the Surreals



Absolute Arithmetic Continuum

Absolute Linear Continuum

A class $(A, <)$ is an *absolute linear continuum* if for all *subsets* of A, L and R where $L < R$, there is always $y \in A$ s.t. $L < y < R$.

Theorem (Ehrlich, 1988 (see [Ehrlich, 2012]))

(NBG) **No** *is (up to isomorphism) the unique real-closed ordered field that is an absolute linear continuum.*

Theorem (Keisler, 1976)

(ZFC+$\exists\kappa$ strongly inaccessible). There exists a unique κ-saturated $\mathbb{R}^*$ such that the latter is isomorphic to **No**.

The results above establish the following conceptual equivalences:

No = Absolute Linear Continuum = κ-saturated field of the hyperreals

PWP-based theories

Recall that:

Part-Whole Principle (PWP)

The whole is greater than its parts:

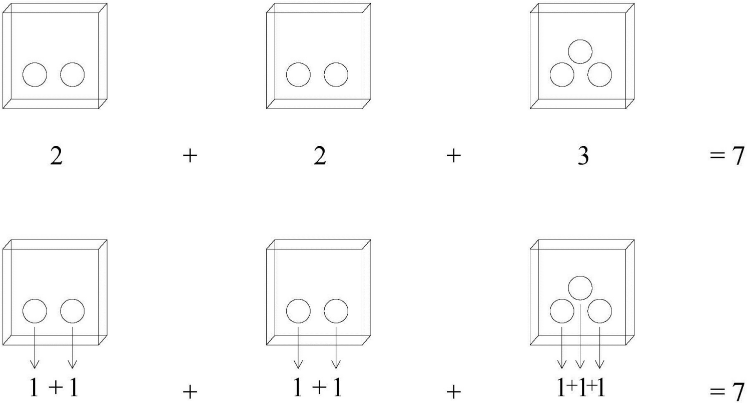
$$X \subset Y \to \mathfrak{s}(X) > \mathfrak{s}(Y)$$

is inconsistent with CP.

Recently, several theories have emerged which obey PWP:

- Katz's theory (contradicts CP)
- Benci-Forti-Di Nasso's theory of *numerosities* (uphold Half-CP)
- Sergeyev's *grossone* (upholds CP?)

Numerosities: Two Ways of Counting



Numerosities: Main Concepts

Labelled Set ([Benci and Di Nasso, 2003], [Benci et al., 2006])

A labelled set A is a pair: $< A, \ell_A >$, where A is the domain, and $\ell_A : A \to \mathbb{N}$, is a function from finite to one.

Throughout, we use the *canonical labelling*: $\ell(n) = n$.

- A 'labelling' produces a decomposition of the set A into its subsets $A = A_0 \cup A_1 \cup A_2, \ldots$, and each $A_n = \{a : \ell_A(a) \leq n\}$.
- Now, the cardinality set $\#A$ is given by the sequence: $< \#A_0, \#A_1, \#A_2, \#A_3, \ldots >$, so that $\#A = \#A_0 + \#A_1 + \ldots \#A_n + \ldots$
- $\#A$ is also called 'approximating function' of A.
- $\gamma_A : \mathbb{N} \to \#A_n$, and $[\gamma_A] = num(A)$ (*numerosity* of A).
- $\mathcal{N} = \{[\phi]_\mathcal{U}$, where $\phi : \mathbb{N} \to \mathbb{N}\}$, and $\mathcal{U}$ is an ultrafilter on the sets of indices of each ϕ.

Numerosities: Euclides Vindicatus

- We can prove that the *numerosity* of the set of *even* numbers (E) is smaller than that of the *natural* numbers ($\mathbb{N}$).
- This is because $num(A) < num(B)$ if and only if, for 'large sets' of n, we have that $\#A_n < \#B_n$.
- Now, $\#E =< 0, 0, 1, 1, 2, ... >$, $\#\mathbb{N} =< 0, 1, 2, 3, 4, ... >$, so for *all* $n > 0$, $\#E < \#\mathbb{N}$, and this is a 'large' set.
- Hence, numerosities satisfy PWP!
- Attempts have been made to extend numerosities to sets of size greater than $\aleph_0$ (see [Benci and Forti, 2020]).

The Euclidean Numbers (and Continuum)

Recently, [Benci and Forti, 2020] has extended the theory of numerosities to a more general theory of the Euclidean numbers:

$$\xi = \sum_{k \in \beta} a_k \ (\textit{with } a_k \in \mathbb{R} \textit{ and } \beta \in \mathit{Ord})$$

Euclidean numbers, overall, make up for a non-archimedean real-closed field $\mathbb{E}$ ('Euclidean continuum') which:

1. Contains all numbers, finite and infinite
2. Contains all *numerosities*
3. Can be defined in the theory ZFC+$\exists\kappa$ strongly inaccessible.

$\mathbb{E}$ is, thus, an *absolute linear continuum* (with numerosities) so we, quite remarkably, have:

No $=$ Absolute Linear Continuum $=$ κ-saturated field of the hyperreals $=$ Euclidean continuum

Grossone: Euclides Vindicatus (again)

- Recently, Sergeyev (see, among others, [Sergeyev, 2017]), has introduced a new measure of infinite cardinalities, which is based on a special number, *grossone* (①).
- ①, like Cantor's ω, comes after *all* natural numbers (but is not the 'least' number after them)
- Moreover, ① can be combined to form other numbers, such as ① $+ 1$, ① $+ 2$, ①$^{-1}$, ①^2, etc.
- ① and its combinations with 'standard' numbers also provide us with measures of infinite sets.
- E.g., the set $\mathbb{N}_{1,2}$, that is, the set: $\{1, 3, 5, \ldots, 1 + 2n, \ldots\}$ has measure ①$/2$. But $\mathbb{N}_{1,2}$ is the set of odd numbers, so measures based on grossone obey PWP!

Grossone: Postulates

Postulates

1. There exist *infinite* and *infinitesimal* numbers, but these arise in the *counting process*, which keeps being *finitary*.
2. Such numbers are not 'new' mathematical *entities*.
3. PWP.

One also has the following fundamental axioms:

Axioms

1. (Infinity). ① exists and is greater than all *finite* natural numbers.
2. (Divisibility). $\mathbb{N}_{k,n} = \{k, k+n, k+2n, ...\}$, has measure ①$/n$.
3. (Identity). ① (algebraically) behaves like any other real number.

Grossone: Assigning Infinite Measures

- The addition of ① (and of its combinations with the other numbers) to the natural numbers produces the extended set:

$$\hat{\mathbb{N}} = \{0, 1, 2, ..., n, n+1, ..., ① - 1, ①, ① + 1, ...\}$$

- $\hat{\mathbb{N}}$ can then be used to assign measures to sets in a way which respects CP
- In particular, one shows that, even if there is a bijection between two sets, X and Y, if $X \subset Y$, then X must, with respect to $\hat{\mathbb{N}}$, be *less than* Y.
- For instance, the set of *odd* numbers has the same Cantorian cardinality as $\mathbb{N}$, but corresponds with only an *initial segment* of $\hat{\mathbb{N}}$ (the segment up to ①/2), so its 'measure' is ①/2.

Bijections and Grossone-Theoretic Measures

Consider the following example in [Sergeyev, 2017], p. 243, of the bijection between natural and odd numbers:

$$\mathbb{N} : 1, 2, 3, 4, 5, ..., ①/2 - 2, ①/2 - 1, ①/2$$

$$\mathbb{O} : 1, 3, 5, 7, 9, ..., ① - 5, ① - 3, ① - 1$$

The example shows that, if we use (initial segments of) the extended set $\hat{\mathbb{N}}$, we may justify the view that grossone provides us with a measure of $\mathbb{O}$ which obeys PWP, but isn't, in principle, incompatible with the 'bijection method'.

Grossone-Theoretic Measures

Using grossone, we get *measures* for specific infinite sets:

1. $\mathbb{N}$ has measure ①.
2. $\mathbb{O}$ has measure ①$/2$.
3. $\mathbb{Z}$ has measure $2$① $+ 1$.
4. $\mathcal{P}(\mathbb{N})$ has measure $2^{①}$.
5. ...

Clearly, all these measures satisfy PWP. Moreover, there are grossone-theoretic measures of the same sets (in the 'Cantorian' sense) which differ from each other:

- Numbers in $[0, 1)$ using 0s and 1s have measure $2^{①}$.
- Numbers in $[0, 1)$ using the decimal system have measure $10^{①}$.

An Axiomatic Theory of Grossone

The theory of grossone may be axiomatised.

E.g., [Lolli, 2015] is an axiomatisation based on:

- PA^2, that is, second-order Peano arithmetic
- The ①-axiom, that is, (Infinity)
- A notion of 'measure' μ (with two additional μ-axioms)
- Lolli introduces a theory, $T^2_{①}$ which contains the μ-measure and, of course, (Infinity).
- As expected, e.g., $T^2_{①} \vdash \mu(\mathbb{N}_{2,2}) = ①/2$.
- Theorem. $T^2_{①}$ is a conservative extension of PA^2.

Open Issues

This shows that the theory of grossone does not add anything, formally, to PA^2 (and is consistent if the latter is).

This fact is consistent with Sergeyev's second postulate (P2 above).

Questions:

- Can the μ-measure be extended to more complex sets? (e.g., sets of rationals, reals, ordinals?)
- What would $ZFC_{①}$ be like?

The Philosophy of Grossone

The ‘philosophy of grossone’ seems to encompass a very peculiar blend of conceptions:

- *(Hilbertian) formalism*: the infinite exists ‘ideally’ (so long we have ‘signs’ to represent it, compare this with *term formalism*)
- *instrumentalism*: a measure of the infinite makes sense as long as it allows us to improve our ‘counting processes’
- *constructivism*: measures of infinite sets (objects) do not exist independently of their descriptions (constructions?)
- *realism*: infinite sets exist

Crucially, it should be noted that the theory of grossone does not aim to define a non-archimedean field along the lines of the non-standard conceptions we have reviewed.

The Continuum Problem

Continuum Problem

How many points are there in the real line?

Cantor's Continuum Problem

1. What is the value of $\mathfrak{c}$?
2. Are there sets of reals of cardinality between that of $\mathbb{N}$ and $\mathbb{R}$?

Cantor's Continuum Hypothesis

$\mathfrak{c} = \aleph_1$.

Theorem (Gödel, Cohen)

CH is independent from ZFC (so, there are models of set theory where CH holds and other where $\neg$CH holds).

A Bunch of Solutions to the Continuum Problem

Cardinality Principle	**Cardinality of the Continuum**
Brouwer's potentialism	denumerably unfinished
Galileo (Peano)	∞
Cantor's Principle	open in ZFC
Vopěnka's Theory	CH holds (?)
No	proper class-sized
Numerosities	depends on labelling
$\mathbb{E}$	strongly inaccessible
Theory of Grossone	depends on descriptions

Concluding Philosophical Remarks

- *Methodological pluralism* (*set-theoretic meta-pluralism*, SMP): there are as many solutions to the continuum problem as approaches to the notion of 'continuum' but: all approaches may be expressible in (re-translated to) a theory of sets:
  - ZC, ZF−P
  - Nelson's IST (ZFC+axioms for 'non-standardness')
  - $ZFC_{①}$ (ZFC+axioms for ①-based measures)
  - ZFC+'∃ a strongly inaccessible cardinal'
  - ...
- *Set-theoretic reductionism* (SR): ZFC+large cardinals may still be held as the foundation of mathematics
- However, (SR)+(SMP) do not entail, *per se*, any solution to any mathematical problem
- Mathematical practice needs (SR) to the extent that it wants to pursue meta-mathematical and foundational investigations

Meta-pluralism

My meta-pluralist might potentially endorse the philosophy expressed by John Conway in [Conway, 1976], p. 66:

> *It seems to us, however, that mathematics has now reached the stage where formalisation within some particular axiomatic set theory is irrelevant, even for foundational studies. It should be possible to specify conditions on a mathematical theory which would suffice for embeddability within ZF (supplemented by additional axioms of infinity if necessary), but which do not otherwise restrict the possible constructions in that theory. Of course the conditions would apply to ZF itself, and to other possible theories that have been proposed as suitable foundations for mathematics (certain theories of categories, etc.), but would not restrict us to any particular theory. This appendix is in fact a cry for a Mathematicians' Liberation Movement!*

Cont'd ([Conway, 1976], p. 66)

Among the permissible kinds of construction we should have:

1. *Objects may be created from earlier objects in any reasonably constructive fashion.*
2. *Equality among the created objects can be any desired equivalence relation.*

But:

- Choice of the axioms is still fundamental
- Set Theory also provides mathematicians with genuinely creative principles
- There's a deeper interplay between practice and SR than conjectured by Conway (cf. [Maddy, 2011])

This Lecture's Main Sources

- [Ehrlich, 2012]
- [Ternullo and Fano, 2021], ch. 3
- [Mancosu, 2016]
- [Sergeyev, 2017]

End

Thanks for attending!

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