Topics in the Philosophy and Foundations of Mathematics

Lecture 5: Axiomatic Set Theory, Independence and New Axioms

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The Paradoxes

Two prominent examples:

Cantor's Paradox

Suppose the 'set of all sets' V exists. Then, one can produce $\mathcal{P}(V)$, the power-set of V. But this must be greater than V, so V contains a set which has *more* sets than itself. $\perp$

Russell's Paradox

Suppose all *properties* collectivise into sets. Now, take the property:

$$\Phi(x) = \{x \notin x\}$$

and the set

$$R = \{x : \Phi(x)\}$$

If the set R exists, then $R \in R \leftrightarrow R \notin R$. $\perp$

The Rise of the Axiomatisation

Reasons to axiomatise set theory:

- Paradoxes
- Widespread use of formalisation
- Hilbert's programme
- Problem of *consistency* of set theory (arithmetic)
- Solution to the *open* problems
- Problem of *foundations*

The process was carried out, at different stages, by: Russell and Whitehead, Zermelo, Skolem, Fraenkel, von Neumann, Gödel, Bernays, and others.[1]

[1] For a comprehensive selection of writings of all these authors, see [van Heijenoort, 1967] and [Grattan-Guinness, 2000].

Zermelo-Fraenkel with Choice (ZFC)

- It was first formulated in [Zermelo, 1908], full axiomatisation is in [Zermelo, 1930], AC was introduced in [Zermelo, 1904]
- Fundamental contributions from: von Neumann (Foundation), Fraenkel and Skolem (Replacement)
- Main variants: ZF (no Choice), ZC (no Replacement), ZF^{-} (no Foundation and Choice), Z (no Choice, Foundation and Replacement), ZF$-$P (no Power-Set and Choice), etc.
- Originally, Separation and Replacement were formulated in second-order logic ([Zermelo, 1930])
- It is considered by most mathematicians as the foundation of mathematics (cf., e.g., [Kunen, 2011])

Extensionality

Axiom of Extensionality

Sets which have the same elements are *equal*.

$$(\forall x)(\forall y)(\forall z)((z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

- So sets are exclusively individuated by their *elements*
- This is taken to be a *basic* property of sets

NB. We shall refer throughout to theories of *pure* sets, ie., those sets which arise from $\emptyset$ (which exists by Separation) and other (set-theoretic) operations; theories of sets with *urelemente* have pure sets + an initial collection of objects.

For all (set-theoretic) purposes one just needs pure sets.

ZU, ZFU, ZFCU, etc. are the corresponding theories *with urelemente*.

Separation

Axiom of Separation

Let F (possibly with free variables $u_1, \ldots, u_n$) be a (first-order expressible) formula. For any set x, there exists the set y, whose elements are in x, which satisfy F.

$$(\forall x)(\exists y)(\forall z)(\forall u_1, \ldots, u_n)(z \in y \leftrightarrow (z \in x \wedge F(z, u_1, \ldots, u_n)))$$

- This is an *axiom schema* (an infinite conjunction of axioms, each given by an instance of F)
- From Separation it follows that there is no Universal Set V (otherwise, by Separation, $R = \{x \in V : x \notin x\}$ would be a set)

Pairing and Union

Axiom of Pairing

There exists a set which contains exactly two elements:

$$(\forall a)(\forall b)(\exists x)((a \in x \wedge b \in x) \wedge (\forall c)(c \in x \rightarrow (c = a \vee c = b))$$

Axiom of Union

There exists the union-set of a set:

$$(\forall x)(\exists y)(\forall u)(u \in y \leftrightarrow (\exists z)(z \in x \wedge u \in z))$$

- Through Pairing and Union one can define several set-theoretic operations

Power-Set

Axiom of Power-Set

There exists the set of all *subsets* of a set:

$$(\forall x)(\exists y)(\forall u)(u \in y \to u \subseteq x)$$

- Through use of the Power-Set Axiom we may define fundamental mathematical concepts: *Cartesian products, relations, functions*, etc.
- Once we have Infinity, we may, among other things, also define: $\mathcal{P}(\omega)$, $\mathcal{P}(\mathcal{P}(\omega))$, etc. (sequence of $\beth$-numbers)

Infinity

Axiom of Infinity

There exists an infinite set.

$$(\exists x)(\emptyset \in x \land (\forall y)(y \in x \to \{y\} \in x))$$

- The set above is called *inductive*
- The existence of an infinite (= inductive) set cannot be proved from the other axioms (it can't even be proved from Replacement!)
- Infinity is equivalent to the existence of ω

Replacement

Axiom of Replacement

If a formula F is a class-function (possibly with a free variable u), then for any set x, $F(x)$ is also a set.

$$(\forall x)(\forall y)(\forall z)(\forall u)(F(x,y,u) \wedge F(x,z,u) \to y = z) \to (\forall X)(\exists Y)(y \in Y \leftrightarrow (\exists x \in X)F(x,y,u))$$

- This is due to Fraenkel and Skolem
- F might have $u_1, \ldots, u_n$ as free variables
- Replacement is fundamental to (Cantorian) set theory: without it, one cannot even prove that $\omega + \omega$ (and $\aleph_1$) exist[2]

[2] The Power-Set Axiom is also needed to prove the existence of uncountable cardinals.

Foundation

Axiom of Foundation

Every non-empty set has an $\in$-minimal element.

$$(\forall x)(x \neq \emptyset \rightarrow (\exists y)\ x \cap y = \emptyset))$$

As a consequence:

- Any set with such characteristics is called *well-founded*
- As a consequence of Foundation: there is no infinite sequence: $x_0 \ni x_1 \ni x_2 \ni \ldots$, there is no set x such that $x \in x$, no cycles are possible: $x_0 \in x_1 \in x_2 \ldots \in x_1 \in x_0$
- It also implies that the universe V (see next few slides) does not contain *non-well-founded* sets ($V = WF$).

Choice

Axiom of Choice

Every family of non-empty sets has a *choice function*.

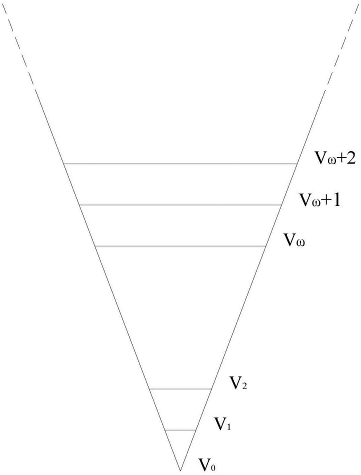
- A family of sets is a collection of subsets of a given set.
- A *choice function* for a family U of subsets of X is an f such that $f(u) \in u$, for each $u \in U$.
- AC is equivalent to Cantor's WOP (Well-Ordering Principle) and Zermelo's Well-Ordering Theorem
- (Countable Choices) = AC restricted to countable sets
- (Dependent Choices) = AC restricted to sets A and relations E, such that: if for all $a \in A$, there exists $b \in A$, such that $E(a, b)$ holds, then there is a (countable) sequence $a_1, \ldots, a_n, \ldots$ such that: $E(a_n, a_{n+1})$, for all $n \in \mathbb{N}$.

Doing set theory (and maths) with AC

The following are results which can be proved using AC (or are equivalent to it):

- Every infinite set has a countable subset.
- The union of a countable collection of countable sets is countable.
- Every vector space has a basis.
- Zorn's Lemma.
- Every set is either finite or has cardinality $\aleph_\alpha$, for some $\alpha \in Ord$.
- König's Theorem (in general, cardinal arithmetic may be carried out satisfactorily only using AC)
- ...

The Universe of Sets



V

The Universe is recursively defined as follows:

$$V_0 = \emptyset$$
$$V_{\alpha+1} = \mathcal{P}(V_\alpha)$$
$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$$

where λ is a limit ordinal.

- All sets are in V
- ZFC does not prove the existence of V, but proves that, for all $\alpha \in Ord$, V_α exists
- V is sometimes taken to be the 'intended model' of set theory
- Each set has a *rank* in V: that is, the least α such that $x \subseteq V_\alpha$.
- The elements of x have each rank *less than* that of x (have appeared at earlier stages). Note that: $x \subseteq V_\alpha \to x \in V_{\alpha+1}$

The Logical (Naive) Conception

- Sets are extensions of properties
- It is taken to imply the Naive Comprehension Principle:

  for any property Φ *there exists the set of the* x*'s which satisfy* Φ

  which, as we know, is inconsistent.
- Classes are now sometimes taken to play the role of *properties* as opposed to sets (= iterative objects)

The Iterative Conception

- The iterative conception is just the conception that sets are formed in stages.
- It's been introduced in [Boolos, 1971], then discussed in [Wang, 1974] and [Parsons, 1977].
- It may be seen as a response to the failure of the 'logical conception'
- In a slogan: sets are the elements of V

Troubles with the Iterative Conception

These are summarised in [Incurvati, 2020], Ch. 3, pp. 70-100 as follows:[3]

- Doesn't seem to account for Naive Comprehension and Extensionality
- The circularity objection
- The no-semantics objection
- The status of Replacement

[3]But these may also be found in other works, such as [Kreisel, 1967], [Boolos, 1971], [Potter, 2004].

Other Axiomatisations

- Class theories: have classes alongside sets (NBG, MK, A)
- Typed theories: distinguish types of sets (STT, RTT, NF, NFU, and others)
- Kripke-Platek (KP): ZF but with Δ_0-Separation and -Replacement, and w/o Power-Set.
- Intuitionistic set theory (IZF, or the weaker CZF)
- 'Small' set theories
- Theories with non-well-founded sets (Aczel's theory, for instance)

Models of Set Theory

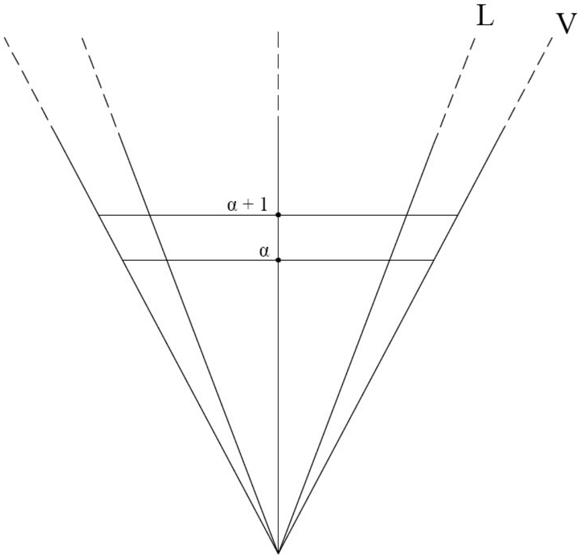
There are two main types:

- Inner models: 'thinnings' of V
- Outer models: 'widenings' of V

Several model-theoretic constructions are employed by set-theorists to produce models of ZFC:

- Constructible universe (Gödel's L)
- Forcing extensions of V
- Elementary embeddings
- Ultrapower constructions
- Further inner models
- Non-standard models (ill-founded models)
- ...

Inner Models: The Constructible Universe



L, cont'd.

Gödel defined the following 'thinning' of V:

$$L_0 = \emptyset$$

$$L_{\alpha+1} = def(L_\alpha)$$

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$$

Rather than taking power-sets at successor stages, one takes the collection of *definable* subsets.[4]

The archetypal inner model L has now given rise to a sequence of strenthenings $L[U]$, which also contain *large cardinals*. This is the *inner model program*.

[4] Definable = defined by a *first-order formula*.

Outer Models: Forcing

- The aim of forcing is to produce models which, in a sense, *extend* V.
- This isn't done literally, though: the 'classic' construction consists in taking a countable model M of the axioms and 'extend' it to another *countable* model N.
- G is the 'object' which is added to M. G should be *generic*, in that it should *avoid* all properties which may be defined by a first-order formula.
- The resulting model $M[G]$ will then have properties which are not true of the initial model (that are not true of V)

The Independence of CH (GCH and AC)

Recall that, given a statement ϕ, ϕ is undecidable from T if $T \nvdash \phi$ and $T \nvdash \neg\phi$.

By the Completeness Theorem, if T has a model M such that $M \models \phi$, then, if T is consistent, then $T \nvdash \neg\phi$ and, vice versa, if T has a model N such that $N \models \neg\phi$, then, if consistent, $T \nvdash \phi$.

Through using L and forcing, it was proved, in two different stages:

Theorem (Gödel, Cohen)

$L \models CH, GCH, AC$. *There exists a forcing extension of* V, $V[G]$, *such that* $V[G] \models \neg CH, \neg GCH, \neg AC$.

So, the combination of both results yields the following:

Theorem (Gödel, Cohen)

ZFC does not prove or disprove CH, GCH. ZF doesn't prove or disprove AC either.

Incompleteness, Again

So ZFC is incomplete: there are genuinely set-theoretic (and even non-set-theoretic) statements that it cannot decide.

- CH.
- GCH.
- (Suslin Hypothesis [SH] = there is no Suslin line).
- 'Every Σ^1_2 set of reals is Lebesgue measurable'.
- Existence of Large Cardinals.

So the fundamental question which arises at this stage is: do we need *new axioms*?

(informal) Poll: Three among the four eminent authors of [Feferman et al., 2000] responded positively.

Criteria for the Introduction of New Axioms

[Gödel, 1947] famously mentioned the following two criteria:

- (int) An axiom is *intrinsically* justified iff it is *derivable from/expresses features of* the concept of set.
- (ext) An axiom is *extrinsically* justified iff it is *rich with (desirable) consequences*

The demarcation between (int) and (ext) is not sharp, though. Cf. my paper with Barton and Venturi ([Barton et al., 2020])

Large Cardinals

The investigation of large cardinals started with the introduction of:

Inaccessible cardinals

A cardinal κ such that κ is: 1) uncountable; 2) regular (that is, not the sum of $< \kappa$-many cardinals, and 3) limit, is called *inaccessible*.

ZFC can't prove that inaccessible cardinals exist. This is because $V_\kappa \models ZFC$, so, otherwise, it would prove its consistency.

The existence of inaccessible cardinals is compatible with the axiom $V = L$.

But there are many which aren't compatible with $V = L$. These are called 'large' large cardinals.

Large Large Cardinals

The 'archetypal' one (measurable cardinal) may be defined using the notion of *elementary embedding*.

Suppose there is an e.e.:

$$j : V \to M$$

If this isn't trivial, then there is α such that $j(\alpha) \neq \alpha$. α is called critical point of j.

The critical point of j above is a *measurable* cardinal.

LCs, cont'd.

A theory T is consistency stronger than U iff

$$Con(T) \rightarrow Con(U)$$

equal iff

$$Con(T) \leftrightarrow Con(U)$$

less strong than iff

$$Con(U) \rightarrow Con(T)$$

It can be proved that all LC axioms are *linearly ordered* as to *consistency strength*.

(ext) LCs have *robust* consequences (although they do not decide CH).

(int) But are they intrinsically justified?

Reflection

Justification for LC's may come from 'abstract motivating principles':

Reflection Principle (RP)

$V \models \phi \leftrightarrow (\exists\alpha) V_\alpha \models \phi^\alpha$

(int) RP admitting of *second-order* formulas and (at most) *second-order* parameters yields all 'small' LC's.

Resemblance Principle

Given α and β, where $\beta > \alpha$, there exists an e.e.:

$$j :< V_\alpha, \in> \rightarrow < V_\beta, \in>$$

(int) Principles such as the above, also involving *classes*, may motivate 'large' large cardinals (cf. [Reinhardt, 1974]).

Determinacy

A set of reals A is determined iff for a game $G(A)$, one of the two players always has a *winning strategy* (the real picked up by either player is in A).

Axiom of Determinacy (AD)

All sets of reals are determined.

- (ext) AD implies *Lebesgue measurability* and the *perfect subset property* for all reals
- AD contradicts AC
- PD is AD for projective sets of reals
- (ext) PD is implied by the existence of a supercompact cardinal
- (int) some intrinsic justification could be derived from (int) of large cardinals (e.g., of *supercompacts*)

Forcing Axioms

Forcing axioms are statements which prescribe the absoluteness of statements between V and 'extensions' of V obtained through specific instances of *forcing*:

- Martin's Axiom: forcing with a c.c.c.-partial ordering
- Martin's Maximum: forcing with partial orderings which preserve stationary subsets of ω_1
- Proper Forcing Axiom: forcing with a *proper* partial ordering
- ...

- (ext) MM, PFA imply $\neg$CH and, in particular, $\mathfrak{c} = \aleph_2$.
- (ext) They also imply PD.
- (int) All FA don't seem to be intrinsically justified.

Axiom of Constructibility

Axiom of Constructibility

$V = L$.

- (ext) it is robustly justified: $V = L$ solves lots of open question in set theory, since

$$L \models CH, GCH, \Diamond, \Box, \neg SH, etc.$$

- (int) it has always been seen as non-intrinsically justified
- (Scott's Theorem, 1967) If there's a measurable cardinal, then $V \neq L$
- if LC's are taken to be intrinsically justified, then (int) of $V = L$ seems to be doomed to failure

The Universe/Multiverse Debate

Universe View

There is just one set-theoretic structure/just one concept of set encompassing all *truths* about sets/from which truths are derivable.

Multiverse View

There are several set-theoretic structures/concepts of set, each of which fixes the realm of set-theoretic truths in different ways.

Arguments in favour of universism

- Platonism about sets
- Incompleteness is due to lack of sharpness of our intuition
- Models are not 'comparable to' V (they are non-standard constructs)
- Categoricity of ZFC_2 (cf. [Kreisel, 1967], [Martin, 2001])
- Internal categoricity ([Button and Walsh, 2018])
- Structural concerns (e.g., V as determinate but 'incomplete structure', cf. [Isaacson, 2011])

Arguments in favour of multiversism

- No new axiom is sufficiently justified
- Experience of models is experience of different concepts of set
- Perspectivism
- Platonic (higher-order) pluralism: models are different platonic entities
- Multiverse theory could be complete (as opposed to the incompleteness of ZFC)
- Foundational considerations (*theoretical virtues*, cf. the debate between myself and Maddy in [Ternullo, 2019])

Summary: Philosophical Questions for the XXI century

- What are the *ultimate* axioms of set theory (mathematics)?
- Is there going to be a final reduction of incompleteness?
- Are there any *absolutely undecidable* questions in mathematics?
- What are *multiverse axioms*?
- What is (if any) the meaning of foundations?
- Do we need any foundations in mathematics?

This Lecture's Main Sources

- [Jech, 2003]
- [Kunen, 2011]
- [Bagaria, 2008]
- [Hamkins, 2020], especially ch. 8
- [Incurvati, 2020]
- [Fraenkel et al., 1973]
- [Ternullo and Fano, 2021], especially ch. 2

End

Thanks for attending!

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