Topics in the Philosophy and Foundations of Mathematics

Lecture 4: Cantorian Set Theory

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Potential vs. Actual Infinite

The distinction between actual and potential infinite dates to Aristotle's *Physics*, III:

- Potential infinite = endless, *incompletable* process (e.g., of counting)
- Actual infinite = entities whose magnitude is *actually greater than* that of any *finite* entity

Set theory thrives on the idea that *actually infinite* entities (= *sets*) can be thought of in a consistent manner

The Euclidean Conception

Historically, one of the major hindrances to the conceivability of *actually infinite* objects lies in Euclid's *Elements*' *Common Notion 5*:

Euclid's Axiom (EA)

The *whole* is greater than its *parts*.

Now, consider the so-called Galileo's Paradox:

There are as many natural as square numbers.

Proof. Take the function $f : \mathbb{N} \to \mathbb{N}$ which associates n to its square n^2. f is 1-1 and onto. □

But EA implies that $\mathbb{N}$ is greater than $\mathbb{S}$ (the set of square numbers).

Contradiction!

Bolzano's and Dedekind's Contributions

Bernhard Bolzano (in [Bolzano, 1851]) was, arguably, one of the first mathematicians to point out that EA should (wholly unproblematically) cease to be valid in the infinite.

Richard Dedekind subsequently introduced the following (fundamental) definition of:

Dedekind-Infiniteness ([Dedekind, 1888])

A set [*System*] S is infinite if and only if S is bijectable with a *proper part* of itself.

But do Dedekind-infinite sets exist?

Dedekind's example is the 'set T of all thoughts'. Given each $t \in T$ let $\phi(t)$ be the thought 't is a thought'. The set of all $\phi(t)$ is a proper part of T, so T is Dedekind-infinite. $\square$

Cantor on the Actual Infinite

In [Cantor, 1883], Cantor discusses three main objections against the a.i.:

- Numbers are used to count, but one can only count finite collections of objects.
- The infinite is bound to 'destroy' the finite.
- If the a.i. exists, then there exist also actual infinitesimals.

Cantor's responses are as follows:

- One can also count infinite collections (= Transfinite)
- There are ways to preserve the meaningfulness of the finite even in the presence of the infinite (consider: '$\omega + 1 \neq 1 + \omega$')
- One can posit the existence of a.i. collections without, at the same time, having to posit the existence of infinitesimals.

Types of Infinity

In the same work and in [Cantor, 1885b], Cantor distinguishes three types of infinity:

- Potential (*improper*): endless processes (= infinitesimals)
- Actual (*proper*): entities greater than the finite ones (*Transfinitum*)
- Absolute: God (*Infinitum Absolutum*)

But he also distinguishes two ways in which the infinite manifests itself:

- *Intra-subjectively* (in the human mind)
- *Trans-subjectively* (in the physical reality)

Examples of the trans-subjective existence of the infinite would be the numbers of *atoms* and *monads* in the physical universe (cf. [Cantor, 1885a]).

Proper and Improper Infinite

As for the mathematical infinite, to the extent that it has found a justified application in science and contributed to its usefulness, it seems to me that it has hitherto appeared principally in the role of a variable quantity, which either grows beyond all bounds or diminishes to any desired minuteness, but always remains finite. I call this infinite the improper infinite [das Uneigentlich unendliche]. [...]

Infinity, in its first form (the improper-infinite) presents itself as a variable finite [veranderliches Endliches]; in the other form (which I call the proper infinite) it appears as a thoroughly determinate [bestimmtes] infinite.

([Cantor, 1883], in [Ewald, 1996], p. 882)

The Absolute

In later work, after the discovery of the paradoxes, Cantor introduced a different characterisation of the Absolute.

He distinguishes between:

- *consistent* multiplicities (have a measure in *Transfinite*)
- *inconsistent* multiplicities (don't have a measure in *Transfinite*)

The former are the *sets*, the latter are *absolutely infinite* collections.

E.g., *Ord* (= class of all ordinals), V (= class of all sets), etc. are all *absolutely infinite*.

The Concept of Set

Cantor never gave a fully transparent (and satisfactory) definition of set (cf. also [Ferreirós, 2010])).

In [Cantor, 1895], he says:

> *By an "aggregate" [Menge] we are to understand any collection into a whole [Zusammenfassung zu einem Ganzen] M of definite and separate objects m of our intuition or our thought. These objects are called the "elements" of M. In signs we express this thus : (1)* $M = \{m\}$.

Troubles with the definition above:

- it seems to presuppose unintelligible 'acts of thinking'.
- it seems to presuppose the existence of *properties* (*intensional* entities).
- it may be prone to paradox.

On the Concept of Set, Again

In a footnote of [Cantor, 1883], Cantor says:

> *In general, by a 'manifold' or 'set' I understand every multiplicity which can be thought of as one, i.e. every aggregate [Inbegriff] of determinate elements which can be united [verbunden] into a whole by some law. I believe that I am defining something akin to the Platonic eidos or idea as well as to that which Plato called miktón in his dialogue 'Philebus or the Supreme Good'. He contrasts this to the apeiron (i.e. the unbounded, undetermined, which I call the improper infinite) as well as to the peras, i.e the boundary; and he explains it as an ordered 'jumble' of both. Plato himself indicates that these concepts are of Pythagorean origin; [...] (in [Ewald, 1996], p. 916)*

Limitation of Size

Cantor's late conception of the Absolute led to the formulation of a doctrine (and corresponding concept of set):

Limitation of size doctrine (LSD)

A multiplicity is a *set* if and only if it isn't 'too big' (that is, if it has a measure in *Transfinite*).

In particular, in Cantor's 1899 letter to Dedekind the doctirne is, more or less articulated as follows:

LSD, again

A property F yields a set iff the *extension* of F is *equinumerous* with an initial segment of *Ord* (collection of all ordinals).

Sets and (Proper) Classes

Limitation of size was subsequently re-used (invoked) by:

- Zermelo in his axiomatisation of Cantor's set theory ([Zermelo, 1908], [Zermelo, 1930])
- [Von Neumann, 1925]'s class-theoretic axiomatisation

Von Neumann's Limitation of Size Axiom (LSA)

A collection is a set iff it is *equinumerous* with V (the universe of all sets).

Von Neumann's theory (later called von Neumann-Bernays-Gödel [NBG]) is the first axiomatic theory based on the *set/class* distinction.

Well-Ordering

Well-ordered set

A set X is said to be *well-ordered* iff it is linearly ordered and every $Y \subseteq X$ has a *least* element.

Cantor took the fact that every set was *well-ordered* or *well-orderable* as a *law of thought*.[1]

He called this the Well-Ordering Principle (WOP).

WOP is equivalent to Zermelo's Axiom of Choice (AC). The Axiom of Global Choice extends WOP also to a.i. collections.

[1] 'In a later article I shall discuss the law of thought that says that it is always possible to bring any well-defined set into the form of a well-ordered set – a law which seems to me fundamental and momentous and quite astonishing by reason of its general validity.' ([Cantor, 1883], in [Ewald, 1996], p. 886).

The Rejection of the Infinitesimals

Cantor formulated several arguments against the introduction of the infinitesimals:

- they are not indispensable
- they are not self-standing entities
- they violate our conception of the linear continuum

As far as the last point is concerned, he purported to prove that, if there are infinitesimal geometric quantities, then a contradiction follows.

But the argument, in fact, already presupposes:

Eudoxus-Archimedes' Axiom

For all $a, b \in \mathbb{R}$, there always exists $n \in \mathbb{N}$ such that $na > b$.

Points-Sets and Derivation

Cantor undertook his investigation of the transfinite as a consequence of the study of: 1) representations of functions through trigonometric series, and 2) point-sets of the continuum. He introduced the notion of:

Derivative of a point-set P

Given a set of points P, its derivative P' is the set of all *accumulation points* of P.

Now, he considered iterations of that operation, and extended it into the infinite: P^{∞}, $P^{\infty+1}$, ... by positing:

$$P^{\infty} = \bigcap_{n \in \mathbb{N}} P^{n}$$

and then one continues: $P^{\infty+1}, P^{\infty+2}, \ldots$.

Very soon he replaced the '∞' symbol with the 'ω' symbol.

The Construction of the Ordinals

Through this, he achieved three fundamental goals:

- Give a definition of *continuous* point-sets.
- Establish a uniqueness criterion for the representability of functions through trigonometric series.
- 'Create' new numbers.

The new numbers soon became the new sequence of transfinite ordinals:

$$\omega, \omega+1, \omega+2, \ldots, \omega+\omega, \omega+\omega+1, \ldots, \omega^{\omega}, \ldots$$

But now the problems arose of:

- how should one interpret such new 'symbols' (entities)
- how one should go about in the construction of the numbers

Ordinals as Order-Types

Cantor's momentous intuition was that each of these numbers could represent the order-type (o.t.) of a given set.

E.g. ω could be seen as the o.t.[2] of the set:

$$\{0, 1, 2, 3, ...\}$$

$\omega + 1$ of:

$$\{1, 2, 3, ..., 0\}$$

$\omega + \omega$ of:

$$\{1, 3, 5, ..., 2, 4, 6, ...\}$$

Note that all of these are *well-ordered* sets.

[2] When transfinite ordinals are used to designate o.t., one uses the symbols: $\overline{\omega}, \overline{\omega + 1}, \overline{\omega + 2}, ...$.

The Generating Principles

As far as the second problem was concerned, in [Cantor, 1883], Cantor introduced three generating principles for the ordinals:

1. (Successor). $s(\alpha) = \alpha + 1$.
2. (Limit) $\lim_{n\to\infty} n = \omega$.
3. (Restriction) Given all ordinals α's of the same cardinality, there exists β after all α's, such that $|\beta| > |\alpha|$.

Through the combined use of the three principles, we may generate all transfinite ordinals.

The (completed) sequence of transfinite ordinals now looks as follows:

$$\omega, \omega + 1, \omega + 2, ..., \omega + \omega, ..., \omega^{\omega}, ...\omega_1, ..., \omega_2, ..., \omega_{\omega}, ...$$

The Cardinality of $\mathbb{R}$

Then Cantor turned to consider the size ('cardinality') of (esp. infinite) sets, and introduced:

Cantor's Principle (also due to Bolzano, Dedekind and Hume (?))

Given any two sets A and B the size of A[a] is equal to the size of B iff there is an $f : A \to B$ s.t. f is 1-1 and onto.

[a] Denoted with $|A|$.

Let $|\mathbb{N}| = N$. If a set has cardinality N, it is said to be *countable*, if greater than N *uncountable*. He was able to prove that:

- $|\mathbb{Z}| = N$ (relative numbers)
- $|\mathbb{Q}| = N$ (rational numbers)
- $|\mathbb{A}| = N$ (algebraic numbers)
- $|\mathbb{R}| > N$ (real numbers)

The Diagonal Proof

Assume, for a contradiction, that $\mathbb{R}$ is countable. Then we could enumerate all real numbers as follows (where a_n is the integer part, the b_{nm}'s are the decimal digits of r_n):

$$r_0 = a_0.b_{00}b_{01}b_{02}..b_{0m}b_{0m+1}..$$

$$r_1 = a_1.b_{10}b_{11}b_{12}..b_{1m}b_{1m+1}..$$

$$r_2 = a_2.b_{20}b_{21}b_{22}..b_{2m}b_{2m+1}..$$

$$\ldots$$

$$r_n = a_n.b_{n0}b_{n1}b_{n2}..b_{nm}b_{nm+1}..$$

$$\ldots$$

The Diagonal Proof/Cont.d

Now define a new real $r_d = a.c_0c_1c_2...$ whose decimal digits c_n take the following values:

$$\begin{cases} c_n = 0 \ \textit{iff} \ \ b_{nn} \neq 0 \\ c_n = 1 \ \textit{iff} \ \ b_{nn} = 0 \end{cases}$$

So, it can be easily seen that $r_d \neq r_n$ for all $n \in \mathbb{N}$. As a consequence, there can be no enumeration of all reals. $\square$

$|\mathbb{R}|$ is, thus, a new cardinality denoted with $\mathfrak{c}$. Now Cantor had two transfinite cardinalities, N and $\mathfrak{c}$.

Were there any others?

The Scale of the Alephs

Cantor soon discovered that the collection of all ordinals 'constructed' using only (Successor) and (Limit) (let us call it Ord_N) was *strictly greater than* N.

Eventually, he put $N = \aleph_0$ and $|Ord_N| = \aleph_1$.

More generally, one has that:

$$|\omega_\alpha| = \aleph_\alpha$$

So, he now had an entire sequence of transfinite cardinalities, which paralleled that of transfinite ordinals:

$$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\omega, \aleph_{\omega+1}, \ldots$$

The Power-Set Principle

Finally, [Cantor, 1891] introduced one further generating principle, the:

Power-Set Operation

Given a set X, its power-set is $\mathcal{P}(X) = \{Y : Y \subseteq X\}$, that is, the collection of all its *subsets*.

Theorem

$|\mathcal{P}(X)| = 2^{|X|}$. *So, in particular,* $\mathfrak{c} = 2^{\aleph_0}$.

Theorem

$|\mathcal{P}(X)| > |X|$.

Proof. See next slide.

One Further Diagonal Proof

Proof. $|\mathcal{P}(X)|$ is, at least, equal to $|X|$, as the function $f : X \to \mathcal{P}(X)$ defined by $x \mapsto \{x\}$ (which maps x to its singleton) is clearly 1-1. Now, consider $Y \in \mathcal{P}(X)$ thus defined:

$$Y = \{x \in X : x \notin f(x)\}$$

Y can't be the image of any $x \in X$, since, if it were, then $x \in Y \leftrightarrow x \notin Y$. Therefore, any, even 1-1, function from X to $\mathcal{P}(X)$ cannot be *onto*, and a simple *diagonal* argument guarantees that Y is *always* definable. □

The $\beth$-numbers

Now, one may define a new hierarchy of numbers:

$$\beth_0 = \aleph_0, \beth_1 = 2^{\beth_0}, \beth_2 = 2^{\beth_1}, \ldots$$

In general, for ordinals α, κ, λ:

$$\beth_{\alpha+1} = 2^{\beth_\alpha}$$

$$\beth_\lambda = \bigcup_{\kappa<\lambda} \beth_\kappa$$

where λ is *limit*.

So, now the natural question arises of what the relationship is between the $\aleph$- and the $\beth$-numbers.

The Continuum Hypothesis

As is known, Cantor hypothesised that:

Continuum Hypothesis

$2^{\aleph_0} = \aleph_1$.[a]

[a] Note that one needs WOP (AC) to formulate CH this way. Without AC, CH just amounts to the statement: 'every subset of the continuum either has cardinality N or $\mathfrak{c}$.'

Years later, Hausdorff formulated the generalised version:

Generalised Continuum Hypothesis

$(\forall \alpha \in \mathit{Ord})\ \beth_\alpha = \aleph_\alpha$.

Post-Cantorian Developments

Problems left unaddressed by Cantorian set theory:

- Concept of 'set'
- Status of CH (and GCH)
- Role of Axioms
- Consistent/inconsistent multiplicity (set/class) distinction
- Role of the Absolute (of classes) in set theory
- Status of set theory
- Actuality of the infinitesimals

This Lecture's Main Sources

- [Cantor, 1932]
- [Hallett, 1984]
- [Dauben, 1979]
- [Lavine, 1994]
- [Ternullo and Fano, 2021], Ch. 1 and 2

End

Thanks for attending!

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