Topics in the Philosophy and Foundations of Mathematics

Lecture 2: Deductivism, Hilbert's Programme and Intuitionism

Claudio Ternullo

May 19, 2022

Universitat de Barcelona

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- Curry's formalism, we have reviewed in the last lecture, may also be seen as a form of deductivism

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- While we are free to select the *axioms* as we please, it is, however, fundamental that the deductive rules are *valid*
- Formal systems do not have *meaning* in themselves, but in relation to the *uses* they fulfill (see next few slides)

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- However, for a structure to be described one can entirely dispense with the nature of the *objects* the structure is supposed to 'contain'.
- Structures may have different *realisations*, what matters the most is *structural invariance*, that is, the collection of properties *preserved* under changes of the relevant domain

Isomorphism

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An isomorphism π is a mapping (function) between the *domains* A and B of two *structures*, $\mathfrak{A}$ and $\mathfrak{B}$, which preserves the *constants, functions and relations*; more precisely, π is an isomorphism if and only if:

- $\pi(c_A) = c_B$.
- $\pi(f_A(a_1, a_2, ..., a_n)) = f_B(\pi(a_1), \pi(a_2), ..., \pi(a_n))$.
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So, mathematics deals with the properties of abstract structures (as exhibited by the axioms) 'up to isomorphism'.

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By the *abstract/concrete* classification, the axioms of Euclidean geometry are the abstract structure, the 'physical reality' is a concrete structure (possibly) instantiating them.

Troubles with Deductivism

As said, deductivism has it that mathematical propositions have meaning insofar as they express (general) properties of abstract systems. This accounts for the *applicability* of mathematics, on the deductivist view: insofar as a *concrete* system has that property, then it *instantiates* a certain set of axioms.

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Responses:

- Hilbert's *finitism* (cf. [Hilbert, 1925], [Hilbert, 1967]).

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- Pure reliance on structural axioms (coupled with appeal to 'physical instantiations') is adequate.

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Note that the Hilbertian Challenge validates 'game formalism', at least as far as infinitary mathematics is concerned.

Finitism and Logic

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PRA (Skolem Arithmetic)

- Logical axioms.
- Derivation rules: MP and variable substitution
- Primitive recursive functions, functions derivable from 'primitive functions' through *recursion rule*.
- Quantifier-free induction.

Cont'd.

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where unbounded quantification is implicit. Do these statements have *meaning*? In a sense, yes, if variables are taken to range over *all* natural numbers not *actually*, but only *potentially*.

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Summary:

1. Finitism: only *finitary* objects are meaningful; sentences involving *infinitary* objects should be interpreted *potentially*.
2. Limitations of classical logic: LEM may cease to be valid.

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2. Retaining LEM (and classical logic) also for infinite collections, but: infinitary mathematics should be interpreted *syntactically* (and *potentially*).

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Ideal elements are, in a sense, 'expansions' or 'generalisations' of real elements.

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- Infinitary mathematics only has an ideal nature; although meaningless in itself, its use is legitimate provided its consistency is shown in *finitary* mathematics
- Mathematics should not be trimmed of LEM (classical logic) or 'Cantor's paradise'; only, the use (and meaning) of these should be re-interpreted.

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- Contrary to Hilbertianism, intuitionism is *revisionary*: classical mathematics should be re-translated, as far as possible, into intuitionistic; otherwise, abandoned.
- Analysis and set theory can be furthered as far as licensed by intuitionistic logic

Revisionism

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Theorem

There exist numbers a^b, where a, b are irrational, which are rational.

Proof. Take $a = b = \sqrt{2}$. If $(\sqrt{2})^{\sqrt{2}}$ is rational, then we're done. If not, take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. a^b is, thus, rational, since $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2$.

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- The number succession is based on the 'two-oneness' of temporal thinking ([Brouwer, 1983])
- Numbers follow each other as Hilbert's number-symbols (*strokes*)

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Responses:

- Kant's views are not subjectivistic: in fact, they do commit themselves to a form of *transcendental* objectivism
- 'Human mind' should be taken to mean 'human thinking', which, perhaps using Husserl's phenomenology, one can construe as being about, again, objective constructs (see [Husserl, 1982], for instance)

Intuitionistic Logic: BHK

Intuitionistic logic relies heavily on *proof* rather than *truth*. Consider *derivation rules* for first-order logic. These are formalised in what is known as Brouwer-Heyting-Kolmogorov (BHK) laws:

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- To prove $\neg A$, prove that a contradiction follows from A
- To prove $A \vee B$, prove either A or B
- To prove $A \wedge B$, prove both A and B
- To prove $A \rightarrow B$, prove that any proof of A can be transformed into a proof of B
- To prove $(\forall x)A(x)$, prove that, for any a, there is a proof of $A(a)$
- To prove $(\exists x)A(x)$, prove that there is an a such that one can prove $A(a)$

Intuitionistic Logic: BHK (consequences)

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- *Reductio* is, thus, no longer a valid argumentative form
- $\neg(\forall x)A(x) \leftrightarrow (\exists x)\neg A(x)$ is invalid, too.

Intuitionistic Potentialism

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which he took to mean that any *potentially* existing number n has the property A.

Intuitionists stress that, in order for this *process* to really take place (be conceptually available), it should run for a *finite* number of time (steps). This is known as *strict potentialism*.

Heyting Arithmetic

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It should be noted that whatever is true in HA, is also true in PA, but the converse does not hold (so HA is a subsystem of PA):

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Theorem (Heyting)

Suppose $\vdash_{HA} \phi$, then ϕ is true in HA (HA proves that ϕ has a realiser).

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To compute $f(a)$ intuitionistically means to approximate $f(a)$ based on a certain amount of information about a which is taken to be sufficient to compute $f(a)$.

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For instance, consider the *blip function*:

$$f(x) = \begin{cases} 1, \textit{ when } x = 0 \\ 0, \textit{ when } x = 1 \end{cases}$$

which has a discontinuity for $x = 0$. This cannot be represented in intuitionistic logic, since, otherwise, it would be true that $(\forall x)(x = 0 \vee \neg x = 0)$, an instance of LEM.

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To summarise:

- Intuitionistic Arithmetic is a subsystem of Classic Arithmetic
- Intuitionistic Analysis is, sometimes, *weaker than*, some others, *just contradicts*, Classic Analysis

Further Remarks

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- [Brouwer, 1983]: 'to be' is 'to be constructed' (Kantian constructivism)
- [Dummett, 1978]: verificationist semantic (classical logic is verification-transcendent)
- [Weyl, 1949], [Feferman, 1987]: intuitionist mathematics (*arithmetic+analysis*) are sufficient to carry out 'practical' maths

This Lecture's Main Sources

- [Linnebo, 2017], Ch. 3, 4, 5.
- [Shapiro, 2000], Ch. 6, 7.
- [Shapiro, 2005], Ch. 8, 9, 10, 11.

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