Fragments of the universal theory of algebraic structures — polynomial reductions

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The purpose of this note is to establish some reductions, operating in polynomial time, between fragments of the universal theory of a given class K of similar algebras, thus arriving at complexity upper bounds of one fragment in terms of complexity of another fragment. Even if our primary interest is in varieties K of residuated lattices, we formulate the problem in general terms to assess the limitations of this method. Parts of the material are generalizations of results already appearing in literature.

Given a class of similar algebras K,

1. $\mathrm{Th}(\mathsf{K})$ is the **first-order** theory of K;
2. $\mathrm{Th}_\forall(\mathsf{K})$ its **universal** theory;
3. $\mathrm{Th}_C(\mathsf{K})$ its **clause** theory;
4. $\mathrm{Th}_Q(\mathsf{K})$ its **quasi-equational** theory; and
5. $\mathrm{Th}_=(\mathsf{K})$ its **equational** theory.

Each of the above fragments is considered as a *decision problem*; that is, a set (such as the equational theory $\mathrm{Th}_=(\mathsf{K})$) to be recognized within some set of possible inputs (such as the set of all identities in the language of K). The set of possible inputs also defines the complement (denoted $\overline{F}$) of a fragment F: in the above, formulas in the corresponding fragment of all formulas of the language that are not valid in K.

For reducing $\mathrm{Th}_\forall(\mathsf{K})$ to $\mathrm{Th}_C(\mathsf{K})$ one need not assume any closure properties for K. The universal theory of K is well known to be fully determined by its clause theory $\mathrm{Th}_C(\mathsf{K})$, and the former is recursive in the latter. Generalizing an approach used in [3] for lattices, we get that $\overline{\mathrm{Th}_\forall(\mathsf{K})}$ is NP-reducible to $\overline{\mathrm{Th}_C(\mathsf{K})}$. This means that the universal theory is definable in the clause theory with the overhead of one polynomially bounded universal quantifier. The case of lattices also shows that, in general, one cannot do better: the clause theory of lattices is in P, as shown in [2], while by results of [3], the universal theory is coNP-complete.

Subsequently, any clause can be easily brought to the form

$$(s_1 = t_1 \wedge \ldots \wedge s_n = t_n) \to (u_1 = v_1 \vee \ldots \vee u_m = v_m)$$

and provided K contains a nontrivial algebra, the antecedent and the succedent can be assumed nonempty. A classic result of McKinsey [5] then allows to polynomially read the validity of a given clause from the validity of at least one of a number of quasi-equations in any K closed under direct products.

Further one would like to (polynomially) reduce $\mathrm{Th}_{\mathrm{Q}}(\mathsf{K})$ to $\mathrm{Th}_{=}(\mathsf{K})$, where possible. To this end, we say that a K has *strong ternary deduction term* (TD term for short) if there is a term $t(x, z, y)$ where y occurs *at most once* and for every algebra $\mathbf{A} \in \mathsf{K}$ and $a, b, c, d \in \mathbf{A}$,

$$t^{\mathbf{A}}(a, a, b) = b$$
$$\text{if } \langle a, b \rangle \in \mathrm{Cg}^{\mathbf{A}}(c, d), \text{ then } t^{\mathbf{A}}(c, d, a) = t^{\mathbf{A}}(c, d, b).$$

It is immediate that varieties with strong TD term have classical TD term (and, therefore, EDPC). Moreover, in varieties with strong TD term, valid quasi-equations polynomially reduce to valid equations. In this respect the behaviour of strong TD terms diverges from that of classical TD terms, which only allow for a recursive transformation of quasi-equations to equations.

We consider these reductions useful as in some cases they beat other methods (such as direct application of the FEP, as given in [1]) for the upper bound provided. E.g., the above mentioned reductions show that the universal theory of the variety of Heyting algebras is in PSPACE (based on the PSPACE upper bound for equational theory, obtained from the sequent calculus), whereas FEP only provides an exponential upper bound.

Also, some of the above reductions might have been used in existing literature. E.g., the coNP-completeness of the universal theory of the variety of Abelian lattice-ordered groups, proved in [8] by looking at width of the partial order of the groups, is immediate from the above results if one takes into account that the variety is generated by the additive group $\mathbb{Z}$ as a quasi-variety. Also, the problem of the complexity of the universal theory (or Q-theory) of Heyting algebras appears to have been considered open: to date, it is claimed unsolved in the Wikipedia entry on Heyting algebras, based on [4], whereas [6] proves doubly exponential time upper bound for the clause theory of Heyting algebras, and eventually [7] gives a method (of independent interest) that proves the universal theory to be PSPACE-complete.

We close by remarking that a polynomial-time reduction from the universal to the equational theory of a class K does not imply a strong ternary deduction term. Indeed for MV-algebras, for example, both the universal and the equational theory are coNP-complete, although the variety does not have ternary deduction term.

References

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