FRAMES, ORDERED ALGEBRAS, AND QUANTIFIERS FOR DEDUCTIVE SYSTEMS

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Relational semantics has proved to be a fundamental tool in the mathematical and philosophical understanding of many non-classical logics, including intuitionistic, modal and substructural logics [2, 4]. Nevertheless, the evolution of the *general* theory of relational semantics is far behind that of algebraic semantics. In this talk we present a first abstract approach to relational semantics, in the spirit of Abstract Algebraic Logic [3]. Even though, for the sake of simplicity, we will confine our discussion to the local aspects of relational semantics, our approach can be extended harmlessly to *arbitrary* propositional logics and, in particular, to global consequences of normal modal logics and to arbitrary substructural logics. We begin by considering a basic question which needs to be addressed by any truly general theory of relational semantics:

A. Can we make precise the idea that a logic has a *local* relational semantics?

To answer this question, we should first make precise what "relational structures" (a.k.a. frames) are in general. The next definitions identify the syntax in which the general notion of a "relational structure" should be expressed. An *ordered language* is an algebraic language $\mathscr{L}$, equipped with an assignment to every basic operation symbol $f \in \mathscr{L}$ of a choice of which arguments of f will be treated as increasing and which ones as decreasing. In this case, given $f \in \mathscr{L}$, we write $f = f(x_1, \dots, x_m; y_1, \dots, y_n)$ to denote the fact that $\vec{x}$ will be treated as increasing and $\vec{y}$ as decreasing. A *labeling map* for an ordered language $\mathscr{L}$ is a function $\beta \colon \mathscr{L} \to \{\Diamond, \Box\}$. A *labeled ordered language* $\mathscr{L}$ is an ordered language $\mathscr{L}$ equipped with a labeling map β.

A *polarity* is a triple $\langle W, J, R\rangle$ where W and J are non-empty sets and $R \subseteq W \times J$. Every polarity $\langle W, J, R\rangle$ induced a Galois connection $(\cdot)^{\rhd} \colon \mathcal{P}(W) \longleftrightarrow \mathcal{P}(J) \colon (\cdot)^{\lhd}$ defined for every $A \subseteq W$ and $B \subseteq J$ as

$$A^{\rhd} := \{j \in J : A \times \{j\} \subseteq R\} \text{ and } B^{\lhd} := \{w \in W : \{w\} \times B \subseteq R\}.$$

From the general theory of Galois connections, it follows that the map $(\cdot)^{\rhd\lhd}$ is a closure operator on W. We denote by $\mathcal{G}(W, J, R)$ the complete lattice of closed sets of $(\cdot)^{\rhd\lhd}$, and denote by $\leq_W$ and $\leq_J$ the relations on W and J respectively defined by the following conditions:

$$w \leqslant_W u \iff u^{\rhd\lhd} \subseteq w^{\rhd\lhd} \quad \text{and} \quad j \leqslant_s lant_J i \iff j^{\lhd} \subseteq i^{\lhd}.$$

We are now ready to introduce the precise definition of a "relational structure". Let $\mathscr{L}$ be a labeled ordered language. An *$\mathscr{L}$-frame* is a structure

$$\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\}\rangle$$

where $\langle W, J, R\rangle$ is a polarity s.t. $\leqslant_W$ and $\leqslant_J$ are partial orders, and for every operation symbol $f \in \mathscr{L}$ s.t. $f = f(x_1, \dots, x_m; y_1, \dots, y_n)$ and $\beta(f) = \Diamond$, we have that $T_f \subseteq W^m \times J^n \times W$ and

(1) For all $\vec{w}_1, \vec{w}_2 \in W^m$, $\vec{j}_1, \vec{j}_2 \in J^n$, and $u_1, u_2 \in W$ s.t. $\vec{w}_2 \leqslant_s lant_W \vec{w}_1$, $\vec{j}_2 \leqslant_s lant_J \vec{j}_1$ and $u_1 \leqslant_s lant_W u_2$,

$$\text{if } \langle \vec{w}_1, \vec{j}_1, u_1\rangle \in T_f, \text{ then } \langle \vec{w}_2, \vec{j}_2, u_2\rangle \in T_f.$$

(2) $\{u \in W : \langle \vec{w}, \vec{j}, u\rangle \in T_f\}$ is a closed set of $(\cdot)^{\rhd\lhd}$ for all $\vec{w} \in W^m$ and $\vec{j} \in J^n$.

If $\beta(f) = \Box$, a dual requirement is asked. We refer to W and J as to the sets of *worlds* and *co-worlds* of $\boldsymbol{F}$.

Let $\mathscr{L}$ be an ordered (labeled) language. An *$\mathscr{L}$-algebra* is a pair $\langle \boldsymbol{A}, \leqslant\rangle$ where $\boldsymbol{A}$ is an algebra s.t. $f^{\boldsymbol{A}}$ is increasing on $\vec{x}$ and decreasing on $\vec{y}$ w.r.t. $\leqslant$, for every connective $f(\vec{x}; \vec{y})$ in $\mathscr{L}$. Every $\mathscr{L}$-frame $\boldsymbol{F}$ induces a *complex algebra* $\boldsymbol{F}^{+}$, which is indeed an $\mathscr{L}$-algebra whose underlying order poset is the complete lattice $\mathcal{G}(W, J, R)$. An $\mathscr{L}$-general frame is a pair $\langle \boldsymbol{F}, A\rangle$ where $\boldsymbol{F}$ is an $\mathscr{L}$-frame and A is the universe of a subalgebra of $\boldsymbol{F}^{+}$. The *complex algebra* $\langle \boldsymbol{F}, A\rangle^{+}$ of $\langle \boldsymbol{F}, A\rangle$ is the substructure of $\boldsymbol{F}^{+}$ with universe A.

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A valuation in a general frame $\langle \boldsymbol{F}, A\rangle$ is a map $v\colon Var \to A$, where Var is fixed countable set of variables. It is possible to define a notion of *satisfaction* at a world w and a notion of *co-satisfaction* at a co-world j of a formula φ (in variables Var) under a valuation v, in symbols $w, v \Vdash \varphi$ and $j, v \succ \varphi$. This allows to associate two consequence relations to every class Fr of $\mathscr{L}$-general frames:

(1) The *local consequence* of Fr, in symbols $\vdash^{l}_{\mathsf{Fr}}$, is defined as follows:

$$\Gamma \vdash^{l}_{\mathsf{Fr}} \varphi \iff \text{ for all } \langle \boldsymbol{F}, A\rangle \in \mathsf{Fr}, \text{ valuation } v \text{ in } \langle \boldsymbol{F}, A\rangle, \text{ and } w \in W, \text{ if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi.$$

(2) The *co-local consequence* of Fr, in symbols $\vdash^{cl}_{\mathsf{Fr}}$, is defined as follows:

$$\Gamma \vdash^{cl}_{\mathsf{Fr}} \varphi \iff \text{ for all } \langle \boldsymbol{F}, A\rangle \in \mathsf{Fr}, \text{ valuation } v \text{ in } \langle \boldsymbol{F}, A\rangle, \text{ and } j \in J, \text{ if } j, v \succ \Gamma, \text{ then } j, v \succ \varphi.$$

The following definition aims to provide a precise answer to question (A): a logic $\vdash$ is an *$\mathscr{L}$-local consequence* if it is the local consequence of some class of $\mathscr{L}$-general frames. It is natural to wonder whether local consequences can be characterized in a transparent way. This is indeed the case, as local consequences turn out to coincide with logics $\vdash$, whose language can be extended to an ordered language $\mathscr{L}$, in such a way that every connective $f(\vec{x}; \vec{y})$ is increasing in $\vec{x}$ and decreasing in $\vec{y}$ w.r.t. the deducibility relation $\vdash$.

Motivated by the above discussion, we began the systematic study of $\mathscr{L}$-local consequences. Among the main results that we obtained we count the following ones:

(1) Every $\mathscr{L}$-local consequence can be associated with a class of *distinguished ordered algebras* $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$, i.e. the class of $\mathscr{L}$-algebras whose principal upsets are deductive filters of $\vdash$.

The fact that the class $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ captures the intuitive idea of an "ordered model of $\vdash$" can be defended both on theoretic and empiric grounds. On the one hand, it turns out that $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ is the class of complex algebras of the $\mathscr{L}$-general frames whose local consequence extends $\vdash$. On the other hand, if $\vdash$ is the local consequence of a normal modal logic (resp. a superintuitionistic logic), then $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ is the corresponding variety of modal (resp. Heyting) algebras equipped with the lattice order.[1]

2. Every $\mathscr{L}$-local consequence $\vdash$ can be associated with a dual logic $\vdash^{\partial}_{\mathscr{L}}$, i.e. the logic preserving degrees of truth [1] of $\{\langle \boldsymbol{A}, \leqslant^{\partial}\rangle : \langle \boldsymbol{A}, \leqslant\rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)\}$.

In the familiar cases, $\vdash^{\partial}_{\mathscr{L}}$ is the expected dual of $\vdash$. For instance, if $\vdash$ is the logic preserving degrees of truth of a variety K of $\mathscr{L}$-algebras with a lattice reduct and ordered under the lattice order, then $\vdash^{\partial}_{\mathscr{L}}$ is the logic determined by the matrices consisting of algebras in K equipped with a lattice ideal.

3. Every $\mathscr{L}$-local consequence can be associated with a class of *distinguished general frames* $\mathsf{Rel}^{\beta}_{\mathscr{L}}(\vdash)$, i.e. the $\mathscr{L}$-general frames whose local and co-local consequences extend respectively $\vdash$ and $\vdash^{\partial}_{\mathscr{L}}$.

Remarkably, $\vdash$ and $\vdash^{\partial}_{\mathscr{L}}$ are, respectively, the local and co-local consequences of $\mathsf{Rel}^{\beta}_{\mathscr{L}}(\vdash)$. Moreover, the classes $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ and $\mathsf{Rel}^{\beta}_{\mathscr{L}}(\vdash)$ are related by a weak duality, which it witnessed by the complex algebra construction and by the behaviour of the co-logic $\vdash^{\partial}_{\mathscr{L}}$.[2]

4. Every $\mathscr{L}$-local consequence $\vdash$ can be semantically extended to a first-order logic $\vdash^{\forall}$ with actualist existential and universal quantifiers, and identity. Remarkably, $\vdash^{\forall}$ can be axiomatized in a very transparent way by means of meta-rules.

References

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[1] More in general, if $\vdash$ is the logic preserving degrees of truth of a variety K of $\mathscr{L}$-algebras with a semilattice reduct and ordered under the meet-order, then $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}})$ coincides with K.

[2] If $\vdash$ is the logic preserving degrees of truth of a variety K of $\mathscr{L}$-algebras with a lattice reduct and ordered under the lattice order, then this weak duality coincides with the one suggested by the theory of canonical extensions.