Logics of variable inclusion and Płonka sums of matrices

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It is always possible to associate with an arbitrary propositional logic $\vdash$, two substitution-invariant consequence relations $\vdash^l$ and $\vdash^r$, which satisfies, respectively, a *left* and a *right variable inclusion constraints*, as follows:

$$\Gamma \vdash^l \varphi \iff \text{there is } \Delta \subseteq \Gamma \text{ s.t. } Var(\Delta) \subseteq Var(\varphi) \text{ and } \Delta \vdash \varphi;$$

$$\Gamma \vdash^r \varphi \iff \begin{cases} \Gamma \vdash \varphi \text{ and } Var(\varphi) \subseteq Var(\Gamma) & \text{or} \\ \Sigma \subseteq \Gamma \end{cases}$$

where Σ is a set of inconsistency terms for $\vdash$ ([7]). Accordingly, we say that the logics $\vdash^l$ and $\vdash^r$ are respectively the *left* and the *right variable inclusion companion* of $\vdash$.

Prototypical examples of variable inclusion companions are found in the realm of three-valued logics. For instance, the left and the right variable inclusion companions of classical (propositional) logic are, respectively, *paraconsistent weak Kleene logic* (PWK for short) [6], and *Bochvar logic* [1].

Recent work [2] linked PWK to the algebraic theory of regular varieties, i.e. equational classes axiomatized by equations $\varphi \approx \psi$ such that $Var(\varphi) = Var(\psi)$. The representation theory of regular varieties is largely due to the pioneering work of Płonka [8], and is tightly related to a special class-operator $\mathcal{P}_{\text{ł}}(\cdot)$ nowadays called *Płonka sums*.

One of the main results of [2] states that the algebraic counterpart of PWK is the class of Płonka sum of Boolean algebras. This observation led us to investigate the relations between left and right variable inclusion companions and Płonka sums in full generality. Our study is carried on in the conceptual framework of abstract algebraic logic [3, 4, 5].

The starting point consists in generalizing the construction of Płonka sums from algebras to logical matrices. This allows us to characterize the matrix models for variable inclusion logics by performing appropriate Płonka sums over direct systems of models of $\vdash$.

As a matter of fact, variable inclusion companions are especially well-behaved in case the original logic $\vdash$ has a specific kind of *partition function* [8, 9], a feature shared by the vast majority of non-pathological logics in the literature. On the one hand, we present a general method to transform every Hilbert-style calculus for a finitary logic $\vdash$ with a partition function into complete Hilbert-style calculi for both $\vdash^l$ and $\vdash^r$. On the other hand, partition functions can be exploited to tame the structure of the matrix semantics $\mathsf{Mod}^{\mathsf{Su}}(\vdash^l)$, $\mathsf{Mod}^{\mathsf{Su}}(\vdash^r)$, given by the so-called Suszko reduced models of $\vdash^l, \vdash^r$. We close our investigation by determining the location of the logics of variable inclusion within the Leibniz hierarchy.

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