Sheaves and Stratification in the Semantics of First-Order Relevance Logics

In the late 1980s, Kit Fine built a sound and complete model theory for first-order relevant logics. The models in Fine's theory are called stratified models. They loosely resemble a hybridization of Kripke models for first-order intuitionistic logic and Routley-Meyer models for zero-order relevant logic. So it's not surprising that stratified models can be given a description in terms of presheaves. But from this perspective, stratified models are a bit unsightly. A sheaf-based semantics seems like it would be preferable. That such a semantics is sound is an immediate corollary of known results. But whether it is complete is an open question.

1. The Category $\mathcal{Z}$

Following Restall, american-style zero-order Routley-Meyer models (hereafter just *models*) are 6-tuples $\langle D, S, N, R, \mathcal{E}^+, \mathcal{E}^-\rangle$ where D is a set called the domain of the model, S is a set whose elements are called *setups*, $N \subseteq S$ is the set of *normal* setups, R is a three-place relation on S, and $\mathcal{E}^+$ and $\mathcal{E}^-$ are functions mapping i-ary predicates to functions from S to 2^{D^i}. Also, if $a \in S$, $m \in D$ and $n \in D$, then a is symmetric in m and n when m and n are not extensionally distinguished at a; that is, when for each i-ary predicate P, $\langle d_1, \ldots, m, \ldots, d_i\rangle \in \mathcal{E}^\pm(P, a)$ iff $\langle d_1, \ldots, n, \ldots, d_i\rangle \in \mathcal{E}^\pm(P, a)$.

Models with domain D are called D-models. If M is a D-model a *variable assignment for M* is a function that maps variables to elements of D. Truth ($\vDash_1$) and falsity ($\vDash_0$) at a setup a in a model M relative to the variable assignment α are defined recursively as follows:

- $M, \alpha, a \vDash_1 Px_1 \ldots x_n$ iff $\langle \alpha(x_1), \ldots, \alpha(x_n)\rangle \in \mathcal{E}^+(P)(s)$.
- $M, \alpha, a \vDash_0 Px_1 \ldots x_n$ iff $\langle \alpha(x_1), \ldots, \alpha(x_n)\rangle \in \mathcal{E}^-(P)(s)$.
- $M, \alpha, a \vDash_1 \neg\phi$ iff $M, \alpha, a \vDash_0 \phi$.
- $M, \alpha, a \vDash_0 \neg\phi$ iff $M, \alpha, a \vDash_1 \phi$.
- $M, \alpha, a \vDash_1 \phi \wedge \psi$ iff $M, \alpha, a \vDash_1 \phi$ and $M, \alpha, a \vDash_1 \psi$.
- $M, \alpha, a \vDash_0 \phi \wedge \psi$ iff $M, \alpha, a \vDash_0 \phi$ or $M, \alpha, a \vDash_0 \psi$.
- $M, \alpha, a \vDash_1 \phi \to \psi$ iff both (i) $Rabc$ and $M, \alpha, b \vDash_1 \phi$ materially imply $M, \alpha, c \vDash_1 \psi$ and (ii) $Rabc$ and $M, \alpha, c \vDash_0 \psi$ materially imply $M, \alpha, b \vDash_0 \phi$.
- $M, \alpha, a \vDash_0 \phi \to \psi$ iff $M, \alpha, b \vDash_1 \phi$ and $M, \alpha, c \vDash_0 \psi$ for some b and c with $Rbca$.

If $Rnab$ for some $n \in N$ then we say $a \leq b$. With mild assumptions on R, we can prove that models satisfy a horizontal heredity principle: if $a \leq b$ and $M, \alpha, a \vDash_{1/0} \phi$, then $M, \alpha, a \vDash_{1/0} \phi$ as well.

Let $M_X = \langle X, S_X, N_X, R_X, \mathcal{E}^+_X, \mathcal{E}^-_X\rangle$ be an X-model and $M_Y = \langle Y, S_Y, N_Y, R_Y, \mathcal{E}^+_Y, \mathcal{E}^-_Y\rangle$ be a Y-model. Let $\mu_D : X \to Y$ and $\mu_S : S_X \to S_Y$ be functions. We say the pair $\langle \mu_D, \mu_S\rangle = \mu$ is a morphism of models (hereafter simply a *morphism*) when

- For all a, b, and c in S_X, $R_X abc$ only if $R_Y \mu_S(a)\mu_S(b)\mu_S(c)$, and
- For all predicates P and $s \in S_X$, $\mu_D(\mathcal{E}^\pm_X(P)(s)) \subseteq \mathcal{E}^\pm_Y(P)(\mu_S(s))$.

Where $\mu_D(\mathcal{E}^\pm_X(P)(s))$ is the obvious extension of μ_D from X to 2^{X^i}. If μ is a morphism, then we say μ is

- R-surjective in the first component when if $R_Y abc$ and $\mu_S(x) = a$, then there are d and e in S_X such that $R_X xde$; $\mu_S(d) = b$ and $\mu_S(e) = c$ (R-surjectivity in the second and third components are defined similarly);
- Totally R-surjective just when μ is R-surjective in all three components;
- Normal when $\mu_S(a) \in N_Y$ iff $a \in N_X$;
- Conservative when $Y \subseteq X$ and for each i-ary predicate, $\mathcal{E}^\pm_Y(P)(\mu_S(a)) = \mu_D(\mathcal{E}^\pm_X(P)(a)) = \mathcal{E}^\pm_X(P)(a) \cap Y^i$.

It should be clear that these properties are preserved under composition. We take $\mathcal{Z}$ to be the category whose objects are models and whose arrows are totally R-surjective, normal, conservative (tnc) morphisms.

2. Sheaf Models and Stratified Models

Recall that if C and $\mathcal{D}$ are categories, then a $\mathcal{D}$-valued C-presheaf is a contravariant functor $C \to \mathcal{D}$. Let $\mathcal{P}$ be the category whose objects are finite subsets of $\mathbb{N}$ and whose arrows are inclusions. To prevent confusion we will write operations on $\mathcal{P}$ using square notation (e.g. $\sqcap$, $\sqsubseteq$, etc.) and write Λ (rather than $\emptyset$) for the bottom element of $\mathcal{P}$. Our interest will be in $\mathcal{Z}$-valued $\mathcal{P}$-preseheaves.

Suppose $\mathcal{F}$ is such a presheaf. For each object $p \in \mathcal{P}$, write $\mathcal{F}(p) = \langle D_p, S_p, N_p, R_p, \mathcal{E}^+_p, \mathcal{E}^-_p \rangle$ and for each arrow $p \sqsubseteq q$ in $\mathcal{P}$, write f^q_p for the corresponding tnc morphism in $\mathcal{Z}$. We will say that $\mathcal{F}$ is nonempty when $D_\Lambda \neq \emptyset$ and say $\mathcal{F}$ is symmetric when for each $p \sqsubseteq q$, $a \in S_p$, $m \in D_p$ and $n \in D_q - D_p$, there is a $b \in S_q$ that is symmetric in m and n such that $f^q_p(b) = a$.

We can use $\mathcal{Z}$-valued $\mathcal{P}$-presheaves to give a semantics for first-order relevance logics. Semantic clauses for the connectives are essentially as above; for quantifiers we use the following 'intuitionistic' clauses:

- $\mathcal{F}(p), \alpha, a \vDash^{\mathcal{F}}_1 \forall x\phi$ iff for every $p \sqsubseteq q$, $m \in D_q$, and $b \in S_q$, if $f^q_p(b) = a$, then $\mathcal{F}(q), \alpha^x_m, b \vDash^{\mathcal{F}}_1 \phi$.
- $\mathcal{F}(p), \alpha, a \vDash^{\mathcal{F}}_0 \forall x\phi$ iff for some $p \sqsubseteq q$, $m \in D_q$, and $b \in S_q$, $f^q_p(b) = a$ and $\mathcal{F}(q), \alpha^x_m, b \vDash^{\mathcal{F}}_0 \phi$.

Where here we take α to be a function from variables to elements of D_p.

Finally, we define a $\mathcal{Z}$-valued $\mathcal{P}$-*sheaf* is a $\mathcal{Z}$-valued presheaf such that for all α and β, if $a \in S_\alpha$ and $b \in S_\beta$ are such that $f^\alpha_{\alpha\sqcap\beta}(a) = f^\beta_{\alpha\sqcap\beta}(b)$, then there is a $c \in S_{\alpha\sqcup\beta}$ so that $f^{\alpha\sqcup\beta}_\alpha(c) = a$ and $f^{\alpha\sqcup\beta}_\beta(c) = b$. A *sheaf-model* is a nonempty symmetric sheaf.

Sheaf models are a particularly natural family of $\mathcal{Z}$-valued $\mathcal{P}$-presheafs to use in a semantic theory. A different, less natural family are the *stratified models*: $\mathcal{F}$ is a stratified model just if $\mathcal{F}$ is a nonempty, symmetric, $\mathcal{Z}$-valued $\mathcal{P}$-presheaf such that for all α and β, if $a \in S_\alpha$ and $b \in S_\beta$ are such that $f^\alpha_{\alpha\sqcap\beta}(a) = f^\beta_{\alpha\sqcap\beta}(b)$, then there is a $c \in S_{\alpha\sqcup\beta}$ so that $a = f^{\alpha\sqcup\beta}_\alpha(c)$ and $b \leq f^{\alpha\sqcup\beta}_\beta(c)$.

With suitable modifications to the underlying zero-order models, one can prove soundness and completeness results for most first-order relevance logics with respect to *stratified* models. Since the class of sheaf models is contained in the class of stratified models, soundness with respect to sheaf semantics is an immediate corollary. Completeness, on the other hand, is an entirely different matter.

3. Plan of the Talk

Since space is limited I will now summarize the remainder of the talk.

After presenting the definitions of sheaf-models and stratified models, I will explain where the completeness proof for the latter breaks when attempting to prove completeness for the former. Following this, we will examine what would be required (essentially a *double* Lindenbaum Lemma) to get around this block. Finally, given time, we will examine an alternative approach via *sheafification*.

The prospects, unfortunately, are not good. Sheaf-models, despite giving a sound and natural semantics, seem to fail to provide a complete semantics for first-order relevance logics. This suggests that perhaps a better way to understand these semantic theories is by examining the geometry of stratified presheaves more generally.