A duality-theoretic approach to MTL-algebras

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In lattice theory, triples constructions date back to Chen and Grätzer's 1969 decomposition theorem for Stone algebras: each Stone algebra is characterized by the triple consisting of its lattice of complemented elements, its lattice of dense elements, and a homomorphism associating these structures. There is a long history of dual analogues of this construction, with Priestley providing a conceptually-similar treatment on duals in 1974 [11] and Pogel also exploring dual triples in his 1998 thesis [10]. Later on, triples decompositions have been extended to account for richer algebraic structures. For example, [9, 1] have provided similar triples decompositions for classes of MTL-algebras, the algebraic semantics for monoidal t-norm based logic, while the works [2] and [3] develop analogue triples representation for classes of residuated lattices. Our aim is to provide a duality theoretic perspective on these recent innovations, showing that the Stone-Priestley duality offers a clarifying framework that sheds light on these constructions.

1 Duality theory for MTL and GMTL-algebras

Since Stone initiated duality theory in logic by presenting his powerful dual categorical equivalence between the category of Boolean algebras and the category of totally disconnected compact Hausdorff spaces having a basis of clopen sets in 1936, his approach has been generalized to wider settings. For example, it has been extended to Priestley duality for distributive lattices, and to structures with various additional operations. For instance, in the setting of residuated structures, we shall mention the works of Urquhart [12], Galatos [7], and Cabrer-Celani [4]. In these works the monoidal operation of the residuated lattices is interpreted with a ternary relation on the dual spaces. In general, this relation may not be the graph of a function. However, under some particular conditions the functionality is guaranteed (see [6, 8]), and this is the case in particular for semilinear residuated lattices.

In order to develop our dual representation of triple constructions, we first improve the cited earlier dualities for residuated structures, and in particular for MTL-algebras. Moreover, we introduce an extended Priestley dual for the algebraic category GMTL of GMTL-algebras (i.e. unbounded MTL-algebras, also known as prelinear semihoops), using the following fact.

Theorem 1. *Let MTL_{div} be the full subcategory of MTL-algebras without zero divisors and SMTL_{ind} be the full subcategory of directly indecomposable pseu-*

docomplemented MTL-algebras. Then the categories MTL_{div}*,* SMTL_{ind}*, and* GMTL *are equivalent.*

2 Duality theory for srDL-algebras

Our main contribution consists in the construction of the extended Priestley duals of srDL-algebras, i.e. MTL-algebras satisfying the following equations:

$$2(x^2) = (2x)^2; \qquad \neg(x^2) \to (\neg\neg x \to x) = 1.$$

srDL-algebras are a large class of MTL-algebras including among others: Gödel algebras, product algebras, the variety generated by perfect MV-algebras as well as the variety generated by perfect MTL-algebras, pseudocomplemented MTL-algebras and the variety of nilpotent minimum algebras without negation fixpoint. Subvarieties of srDL-algebras enjoy the representation in triples shown in [1], i.e. they are categorically equivalent to a category whose objects are triples made of a Boolean algebra, a GMTL-algebra, and an operator joining them.

We prove that their dual space can be constructed using the Stone space associated to their Boolean skeleton and the extended Priestley dual of their radical (i.e., the intersection of maximal filters) via a *rotation construction* that is similar to the one used in [5], named *reflection construction*, to prove a dualized version of the categorical equivalence present in the same paper between bounded Sugihara monoids and Gödel algebras enriched with a Boolean constant.

Moreover, we prove a dual equivalence between the algebraic category of srDL-algebras and suitably defined *dual quadruples* made of a Stone space, and extended Priestley dual of a GMTL-algebra and two operators associating them.

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