Distributivity in logic and algebra

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Abstract

In a general logical context, the so-called Proof by Cases Property seems to be the most distinctive property of disjunction [2]:

$$\text{PCP:}\quad \text{If } \Gamma, \varphi \vdash \chi \text{ and } \Gamma, \psi \vdash \chi, \text{ then } \Gamma, \varphi \vee \psi \vdash \chi.$$

Indeed, this property holds in classical logic as well as in many substructural logics, e.g., in intuitionistic logic, MTL, and FL_{ew}. In particular, it is at the heart of the Disjunction Elimination rule in Gentzen's Natural Deduction calculus [3]:

$$\vee E : \frac{\varphi \vee \psi \quad \begin{matrix}[\varphi] \\ \chi\end{matrix} \quad \begin{matrix}[\psi] \\ \chi\end{matrix}}{\chi}.$$

Given a lattice L, one can always define the so-called *order logic* in the following way:

$$\Gamma \vdash \varphi \text{ iff for any evaluation } e \text{ on } L, \textstyle\bigwedge_{\gamma \in \Gamma} e)(\gamma) \leq e(\varphi).$$

Then, in the context of an order logic with lattice conjunction $\wedge$ and disjunction $\vee$, PCP implies distributivity of $\wedge$ w.r.t. $\vee$.

However, notice that PCP only involves the disjunction operator $\vee$. Accordingly, a natural question is which notion of distributivity arises in an order logic with only $\vee$. In algebraic terms, the question translates to which notion of distributivity arises in a $\vee$-semilattice that agrees with PCP.

We have examined different notions of distributivity for semilattices that appear in the literature. In particular, Grätzer and Schmidt's notion of 1962 (see [4] and [5]) is the usual and strongest one, not allowing for finite examples that are not lattices. In order of strenght, we prove that the notions we have found behave as follows:

$$(\mathbf{GS}) \Rightarrow (\mathbf{K}) \Rightarrow (\mathbf{H}) \Rightarrow (\mathbf{B}) \cdots \Rightarrow (\mathbf{S_n}) \Rightarrow \cdots (\mathbf{S_3}) \Rightarrow (\mathbf{S_2}),$$

where $\Rightarrow$ means implication and the abbreviations correspond to the notions introduced by the already mentioned Grätzer and Schmidt in [4], Katriňák in [6], Hickman in [7], Balbes in [1], and Schein in [9], respectively. We provide the corresponding countermodels in order to see that they are not equivalent. We also prove, when it is the case, that finite countermodels do not exist. It is also the case that the semilattice version of the notion introduced in [8] for posets is equivalent to the notion **(H)**.

Finally, we show that the notion that agrees with PCP is **(H)**.

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