Decidability for S5 Gödel logics

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Many-valued modal logics are defined via a many-valued Kripke semantics with propositional connectives interpreted locally at each world in some algebra, where the accessibility relation is either crisp (Boolean-valued) or takes arbitrary values in the algebra. In this work we study many-valued modal logics extending propositional Gödel logic G, defined as the logic (with designated value 1) of the algebra $\mathbf{G} = \langle [0,1], \min, \max, \to_{\mathsf{G}}, 0, 1 \rangle$, where

$$a \to_{\mathsf{G}} b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise.} \end{cases}$$

As well as being an important many-valued logic, G has also been studied as an intermediate logic, axiomatized by extending intuitionistic logic with the axiom of prelinearity $(\phi \to \psi) \lor (\psi \to \phi)$ [4].

The semantics for Gödel modal logics are provided by many-valued Kripke models $\langle W, R, V \rangle$ where W is a non-empty set of worlds, $R \colon W \times W \to [0,1]$ is a many-valued accessibility relation, and V is a function mapping a formula ϕ at a world x to a value $V(\phi, x)$ in $[0,1]$ with propositional connectives interpreted in the algebra $\mathbf{G}$ and modalities interpreted according to the clauses

$$V(\Box\phi, x) = \bigwedge\{Rxy \to_G V(\phi, y) \mid y \in W\}$$
$$V(\Diamond\phi, x) = \bigvee\{\min\{Rxy, V(\phi, y)\} \mid y \in W\}.$$

Recently in [2] it has been shown that validity in the logic $\mathsf{K}(\mathbf{G})$ of all such many-valued Kripke models is PSPACE-complete. Similarly, validity in the logic $\mathsf{K}(\mathbf{G})^{\mathsf{C}}$ of all crisp models, i.e., those models such that $Rxy \in \{0,1\}$ for all $x, y \in W$, is also PSPACE-complete.

In this work we consider modal Gödel logics based on "S5" many-valued Kripke models; that is, models where the accessibility relation R satisfies the many-valued analogues of reflexivity, $Rxx = 1$ for all $x \in W$, symmetry, $Rxy = Ryx$ for all $x, y \in W$, and transitivity, $\min\{Rxy, Ryz\} \leq Rxz$ for all $x, y, z \in W$. The logic of these models we denote by $\mathsf{S5}(\mathbf{G})$. The logic of crisp models, where R is a equivalence relation, we denote by $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$.

In [2] it was shown that validity in $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ is co-NP complete. The non-crisp version $\mathsf{S5}(\mathbf{G})$ was not considered, however. In this work we establish the following result.

Theorem. *Validity in* $\mathsf{S5}(\mathbf{G})$ *is decidable, indeed co-NP-complete.*

Since the finite model property fails for the given Kripke semantics for $\mathsf{S5}(\mathbf{G})$, our proof of this theorem makes use of an alternative semantics introduced in [2] that does admit a finite model property. The key idea of this semantics is to restrict the evaluation of modal formulas to a specific set of values. Although the proof can be given completely using frames, an algebraic approach is more convenient. Algebraic semantics for $\mathsf{S5}(\mathbf{G})$ are provided by monadic Heyting algebras satisfying the prelinearity law $(x \to y) \vee (y \to x) \approx 1$, reflecting the fact that $\mathsf{S5}(\mathbf{G})$ is axiomatized by extending MIPC (corresponding to the one-variable fragment of first-order intuitionistic logic) with the prelinearity axiom [3]. Similarly, $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ is axiomatized by adding to $\mathsf{S5}(\mathbf{G})$ the constant domain axiom $\Box(\Box\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$ [5]. By associating a corresponding prelinear monadic Heyting algebra with each frame from the standard and the alternative semantics, we establish a finite model property for the varieties of these algebras, yielding also the desired decidability and complexity results.

Moreover, the algebraic approach offers another nice connection. By [1], there exists a one-to-one correspondence between prelinear monadic Heyting algebras $\langle \mathbf{A}, \Box, \Diamond \rangle$ and pairs of prelinear Heyting algebras $\langle \mathbf{A}, \mathbf{A}_0 \rangle$ where $\mathbf{A}_0$ is a subalgebra of $\mathbf{A}$ satisfying a property called relative completeness. It turns out that this subalgebra $\mathbf{A}_0$ plays a similar role in restricting the modal values as the alternative semantics.

References

[1] G. Bezhanishvili. Varieties of monadic heyting algebras - part I. *Studia Logica*, 61(3):367–402, 1998.

[2] X. Caicedo, G. Metcalfe, R. Rodríguez, and J. Rogger. Decidability in order-based modal logics. *Journal of Computer System Sciences*, 88:53–74, 2017.

[3] X. Caicedo and R. Rodríguez. Bi-modal Gödel logic over [0,1]-valued Kripke frames. *Journal of Logic and Computation*, 25(1):37–55, 2015.

[4] M. Dummett. A propositional calculus with denumerable matrix. *Journal of Symbolic Logic*, 24:97–106, 1959.

[5] P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer, Dordrecht, 1998.