On the coextension of cut-continuous pomonoids

David Kruml[1], Jan Paseka[1] and Thomas Vetterlein[2]

[1] Department of Mathematics and Statistics
Masaryk University, Brno, Czech Republic
{kruml, paseka}@math.muni.cz
[2] Department of Knowledge-Based Mathematical Systems
Johannes Kepler University Linz, Linz, Austria
Thomas.Vetterlein@jku.at

Abstract

A map between posets is residuated if the inverse image of each principal ideal is again a principal ideal. A generalisation of this property has been studied in a number of papers: the so-called *cut-continuous maps* can be defined by the condition that the inverse images of principal ideals are just cuts, that is, sets consisting of the lower bounds of all their upper bounds [1]. The idea underlying the present work is to apply this definition in the context of pomonoids. For a pomonoid to be residuated means that the multiplication from the left or from the right is a residuated map. We relax this definition, just requiring that multiplication from the left or right is cut-continuous.

Cut-continuous pomonoids thus comprise residuated posets. Our actual interest concerns the latter structures and the starting point of considerations is the following. Let L be a commutative, integral residuated lattice and let C be a filter, that is, an upwards closed subalgebra of L. Then C induces a congruence θ_C of L and in fact all congruences of L are of this form. Let $P = L/\theta_C$ be the quotient; then we call L a *coextension of P by C*.

Given commutative, integral residuated lattices P and C, we raise the question how to determine the coextensions of P by C. It should be understood that this question is feasible only if C is assumed to be of a particularly simple structure. The typical example is $C = \mathbb{R}^-$, the negative real cone endowed with the natural order and usual addition. In order to determine coextensions of the mentioned type, we may furthermore adopt what we could call a local viewpoint: we consider the monoidal multiplication of the enlarged structure restricted to single congruence classes. More specifically, let L be as above, possessing the filter C. Then each congruence class is, first of all, a C-module: it is a lattice on which the elements of the residuated lattice C act, and the action is in both arguments residuated. Second, let R, S be two congruence classes such that $R = S \to (R \cdot S)$. Then the mapping $R \times S \to T,\ (r, s) \mapsto r \cdot s$, that is, the monoidal multiplication restricted to $R \times S$, is separately in each argument residuated and action preserving; we speak of a bimorphism of C-modules.

These simple observations imply that in order to construct a coextension of P by C, we have to associate with each element of P a C-module, and we have to determine bimorphisms in line with the product on P. The first step, the systematic determination of C-modules, does not seem possible in general. For a specific C, however, the possibilities can be very limited, an example being the mentioned negative real cone. But

second, once the C-modules are given, the question is how to explore the bimorphisms. This is where the present paper intends to contribute. An application is the *construction of residuated structures related to fuzzy logics*, in particular left-continuous t-norms [4].

Given modules R, S, T over some cut-continuous pomonoid C, we are interested in determining all the bimorphisms $\psi\colon R \times S \to T$. A tensor product facilitates this task, allowing us to focus on morphisms instead.

Definition 1. Let C be a cut-continuous pomonoid. Then a *C-module* is a poset R together with an *action* $\star\colon C \times R \to R$ such that (i) $a \star (b \star x) = (a \cdot b) \star x$ and $1 \star x = x$ for any $a, b \in C$ and $x, y \in R$ and (ii) $\star$ is separately cut-continuous.

A *morphism* between C-modules R and S is a cut-continuous map $\varphi\colon R \to S$ such that $\varphi(a \star x) = a \star \varphi(x)$ for any $a \in C$ and $x \in R$. Finally, for C-modules R, S, and T, a *bimorphism* is a map $\psi\colon R \times S \to T$ that is a morphism of C-modules in each argument. Let C be a cut-continuous pomonoid and let R, S be C-modules. By a *tensor product* of R and S, we mean a bimorphism π from $R \times S$ to a further C-module denoted by $R \otimes_C S$, such that for any bimorphism ψ from $R \times S$ to a C-module and sup-lattice T there is a morphism $\tilde{\psi}\colon R \otimes_C S \to T$ such that $\psi = \tilde{\psi} \circ \pi$.

Theorem 1. *Let C be a cut-continuous pomonoid and let R, S be C-modules. Then there is a tensor product $R \otimes_C S$ of R and S.*

The results of the present work represent, in our opinion, a step in the direction of providing a framework for the determination of coextensions of cut-continuous pomonoids. Several problems remained open and should be addressed in further research. E.g., our results apply to the construction of coextensions in the case of a total order, whereas the general case remains tricky; the problem is that the restriction of the monoidal operation to a pair of congruence classes is seen to be cut-continuous only under the assumption of linearity. Some important results in this direction were obtained in [2]. Note also that finite residuated lattices form another important class satisfying the assumptions for the restriction.

Acknowledgement. The research was supported by the project "New approaches to aggregation operators in analysis and processing of data" Nr. 18-06915S financed by the Czech Grant Agency (GAČR) .

References

1. M. Erné. Tensor products for bounded posets revisited. *Order* 7:295–314, 1990.
2. M. Erné, Cut-continuity, cut-compatibility, filters and congruences of pomonoids. preprint, 2018.
3. M. Erné, J. Picado. Tensor products and relation quantales. *Algebra Universalis* 78: 461–487, 2017.
4. T. Vetterlein. Totally ordered monoids based on triangular norms. *Communications in Algebra* 43:2643–2679, 2015.