On principal congruences in distributive lattices with a commutative monoidal operation and an implication

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We introduce and study a variety of algebras that properly includes the varieties of integral distributive commutative residuated lattices (see [6, 7]) and of weak Heyting algebras (see [3]). We call the members of this variety *distributive lattices with a commutative monoidal operation and an implication*, and we denote the variety by DLCMI.

It is very convenient to have good descriptions of the principal congruences of the algebras of a variety. One type of description is having first-order definable principal congruences. A much simpler and useful type of description is having equationally definable principal congruences (EDPC). This concept was introduced in [4, 5]. The property EDPC is also of logical interest because an algebraizable logic whose equivalent algebraic semantics is a variety has some form of deduction-detachment theorem if and only if the variety has EDPC. There are varieties that do not have EDPC but where it is still possible to have a good characterization of the principal congruences with the following local version of EDPC:

- There exists a finite family of quaternary terms $\{u_{(i,n,k)}, v_{(i,n,k)}\}_{i=1}^{r}$ (with $n, k \geq 0$) such that for every principal congruence $\theta(a, b)$ of any algebra A in the variety it holds that $(c, d) \in \theta(a, b)$ if and only if there exist $n, k \geq 0$ such that $u_{(i,n,k)}(a, b, c, d) = v_{(i,n,k)}(a, b, c, d)$ for every $i = 1, \ldots r$.

We say that a variety of algebras has *locally equationally definable principal congruences* if there exists a finite family of quaternary terms such that in every algebra of the variety they define the principal congruences in the way just described. In particular, the variety of integral commutative distributive residuated lattices has locally equationally definable principal congruences (it can be deduced from a more general result given in [1], see also [2]) and the variety of weak Heyting algebras has this property too, which it was proved in [8].

Our main goal is to show that DLCMI has locally equationally definable principal congruences. We apply the description of principal congruences in order to study compatible functions.

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