Failures of Coherence

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A variety $\mathcal{V}$ is said to be *coherent* if every finitely generated subalgebra of a finitely presented member of $\mathcal{V}$ is again finitely presented. This notion has been studied quite widely in algebra (in particular, in connection with sheaves, rings, groups, and semigroups) and was formulated in a model-theoretic setting by Wheeler [8], who proved that coherence of $\mathcal{V}$ is implied by, and indeed in conjunction with amalgamation and another property implies, the existence of a model completion for the first-order theory of $\mathcal{V}$.

We prove in [5] that coherence forms a key ingredient of *uniform interpolation*. Uniform interpolation was established for intuitionistic propositional logic IPC by Pitts [6] and used by Ghilardi and Zawadowski [3] to prove that the first-order theory of Heyting algebras has a model completion. As shown by van Gool et al. [4], uniform interpolation properties for a variety $\mathcal{V}$ may be defined using equational consequence in $\mathcal{V}$ and related to properties of compact congruences on free and finitely presented algebras of $\mathcal{V}$. In particular, if $\mathcal{V}$ has the amalgamation property, then it has right uniform deductive interpolation if and only if the restriction of any compact congruence on a finitely generated free algebra of $\mathcal{V}$ to a free algebra over a subset of the generators is again compact. We show here that this latter condition is equivalent to coherence.

Following Pitts' theorem for IPC, many proofs of uniform interpolation or its failure for various logics have appeared in the literature. In particular, all intermediate logics with Craig interpolation admit uniform interpolation, but some modal logics, including S4 and K4, admit Craig interpolation but not uniform interpolation [3]. For the modal logic K, extra care is needed. A semantic proof of uniform interpolation was given for K (also Gödel-Löb logic GL and Grzegorgczyk logic S4Grz) by Visser in [7], and a Pitts-style proof was provided (also for KT) by Bílková in [2]. However, these proofs establish a uniform "implication-based" interpolation property, and not the uniform deductive "consequence-based" interpolation property studied in [4]. The same observation applies to uniform interpolation results for substructural logics (varieties of residuated lattices) established by Alizadeh et al. in [1].

We provide in [5] a general criterion for establishing the failure of coherence, and hence also of uniform deductive interpolation, for varieties generated as a prevariety by a class of algebras that admit a term-definable reduct consisting of a complete semilattice with an order-preserving unary operation $\Box$. More precisely, we show that the existence of such a term-definable reduct guarantees that either coherence fails or a certain n-potency equation is satisfied for some $n \in \mathbb{N}$. We use this criterion to show that any coherent variety of Boolean algebras with operators that is closed under canonical extensions has equationally definable principal congruences. In particular, K is not coherent, does not admit uniform deductive interpolation, and its first-order theory does not have a model completion. Indeed, the same is true of any normal modal logic closed under canonical extensions for which $\boxdot^n x \approx \boxdot^{n+1} x$ fails for all $n \in \mathbb{N}$ (where $\boxdot x = \Box x \wedge x$). We obtain similar results also for varieties of double-Heyting algebras, residuated lattices, and lattices.

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