Undecidability of $\mathsf{FL_e}$ in the presence of structural rules

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The decidability of the equational and quasi-equational theories for the variety of commutative residuated lattices ($\mathcal{CRL}$) axiomatized by $\{\cdot, 1, \leq\}$-inequalities has been fully classified. It was shown by van Alten (2005) that quasi-equational theories axiomatized by *knotted inequations* (k_n^m) (i.e. universally quantified inequations of the form $x^n \leq x^m$ for $n \neq m$) are not only decidable, but enjoy the finite embedability property (FEP). In fact, Galatos and Jipsen (2013) showed that $\mathcal{CRL} + (\mathrm{k}_n^m) + \Gamma$ has the FEP for any set Γ of $\{\cdot, 1, \vee\}$-equations. Viewed proof-theoretically, these results show that deducibility in the Full Lambek calculus with exchange ($\mathsf{FL_e}$) axiomatized by knotted inference rules are decidable.

Dropping commutativity (i.e., exchange), Horčík (2015) showed that $\mathcal{RL} + (\mathrm{k}_n^m)$ has an undecidable word problem for any $1 \leq n < m$ or $2 \leq m < n$, which implies deducibility is undecidable in the corresponding logic. By utilizing the theory of residuated frames, as developed by Galatos and Jipsen (2013), and an encoding based on counter (or Minsky) machines with an undecidable halting problem, Horčík constructs a residuated lattice $\mathbf{R}$ in $\mathcal{RL} + (\mathrm{k}_1^2) + (\mathrm{k}_3^2)$ such that for any variety $\mathcal{V} \subseteq \mathcal{RL}$, $\mathbf{R} \in \mathcal{V}$ implies $\mathcal{V}$ has undecidable word problem. Bootstrapping this result via a deduction theorem, Chvalovský and Horčík (2016) showed that the equational theory of $\mathcal{RL} + (\mathrm{k}_n^m)$ is undecidable for any $1 \leq n < m$, and thus provability in the corresponding knotted extension of FL is undecidable. As a consequence, these results capture many non-commutative varieties satisfying equations in the signature $\{\cdot, 1, \vee\}$. For example, $\mathcal{RL} + (x \leq x^2 \vee x^3)$ has an undecidable word problem since it contains $\mathbf{R}$.

However, in general these results are not true in the commutative case, as it would contradict van Alten's result. With the exception of knotted rules, little is known about the decidability questions for varieties of commutative residuated lattices axiomatized by equations in the signature $\{\cdot, 1, \vee\}$, e.g., are the (quasi)equational theories for $\mathcal{CRL} + (x \leq x^2 \vee x^3)$ or $\mathcal{CRL} + (xy \leq x^2y \vee x^3y^2 \vee 1)$ decidable?

With a method similar in spirit to the above, the present work defines an infinite set D of $\{\cdot, 1, \vee\}$-equations where for any finite $\Gamma \subset D$ there exists $\mathbf{R}_\Gamma \in \mathcal{CRL} + \Gamma$ such that for any variety $\mathcal{V} \subseteq \mathcal{RL}$, $\mathbf{R}_\Gamma \in \mathcal{V}$ implies $\mathcal{V}$ has an undecidable word problem. This is achieved by encoding the computation of a Minsky machine with an undecidable halting problem in the language of commutative semirings (where the role of $\vee$ is essential), and the soundness of the encoding is proved using the theory of residuated frames. Furthermore, there is an infinite proper subset $D_\epsilon \subset D$ such that the above can be extended to undecidability of the equational theory via a deduction theorem if $\mathcal{V} \subseteq \mathcal{CRL} + (\mathsf{d})$ for some $(\mathsf{d}) \in D_\epsilon$. Membership of D corresponds to whether there exists positive solutions to certain systems of linear equations and, in a sense, captures most rules in the signature. For instance, $(x \leq x^2 \vee x^3) \in D_\epsilon$ and $(xy \leq x^2y \vee x^3y^2 \vee 1) \in D \setminus D_\epsilon$. However, the decidability of certain equations remain unknown in $\mathcal{CRL}$, e.g., $x \leq x^2 \vee 1$ is neither in D nor does it imply a knotted rule, but we can fully specify the class of such rules in a canonical way.

These results imply undecidability for deducibility and provability in the corresponding structural extensions of $\mathsf{FL_e}$, e.g., we show $\mathcal{CRL} + (x \leq x^2 \vee x^3)$ has an undecidable equational theory, and thus provability in $\mathsf{FL_e} + (\mathsf{d})$ is undecidable, where

$$\frac{\Gamma, \Delta, \Delta, \Pi \Rightarrow \phi \quad \Gamma, \Delta, \Delta, \Delta, \Pi \Rightarrow \phi}{\Gamma, \Delta, \Pi \Rightarrow \phi}\ (\mathsf{d}).$$