A poset product-representable family of BL-algebras*

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The variety $\mathcal{BL}$ of BL-algebras is the algebraic counterpart of BL (Hájek's **B**asic Fuzzy **L**ogic [4]). A BL-algebra is a divisible and commutative residuated lattice which is also prelinear. Among others, the varieties of MV-algebras and product algebras are well-known subvarieties of $\mathcal{BL}$. Due to the prelinerity property, the fundamental structures in the study of $\mathcal{BL}$ are its totally ordered members (BL-chains). Focused on BL-chains, it was proved that they can be completely described as an ordinal sum (of simpler structures).

Theorem 1 (Subdirect representation theorem. See [4])**.** *Each BL-algebra is a subdirect product of BL-chains.*

Theorem 2 (Decomposition theorem. See [1])**.** *Each non-trivial BL-chain admits (up to isomorphism) a unique decomposition into an ordinal sum of non-trivial totally ordered Wajsberg hoops.*

Since every BL-algebra can be embedded into the direct product of BL-chains and every BL-chain can be decomposed as an ordinal sum, Jipsen and Montagna proposed in [6] a construction called poset product as a sort of generalization of direct product and ordinal sum. Briefly, the poset product is a subset of a direct product which is defined by using a partial order over the index set. Specifically,

Definition 3. *Let $P = \langle P, \leq \rangle$ be a poset and let $\{\mathbf{A}_p : p \in P\}$ be a collection of BL-algebras sharing the same neutral element* 1 *and the same minimum element* 0*. The* poset product $\bigotimes_{p\in P} \mathbf{A}_p$ *is the residuated lattice $\mathbf{A} = \langle A, \cdot, \to, \vee, \wedge, \bot, \top \rangle$ defined as follows:*

1. *The domain of $\mathbf{A}$ is the set of all maps x belonging to $\prod_{p\in P} A_p$ such that for all $p \in P$, if $x_p \neq 1$, then $x_q = 0$ for all $q > p$.*
2. *$\bot \equiv 0$ (the constant map 0) and $\top \equiv 1$.*
3. *The monoid operation and the lattice operations are defined pointwise.*
4. *The residual is as follows:*

$$(x \to_{\mathbf{A}} y)_p = \begin{cases} x_p \to_{\mathbf{A}_p} y_p & \text{if } x_q \leq_q y_q \text{ for all } q < p; \\ 0 & \text{otherwise.} \end{cases}$$

*Joint work with Manuela Busaniche.

In [3], based on the results of [5, 6, 7], it is shown that every BL-algebra can be thought as a subalgebra of the poset product of a collection of BL-chains.

Theorem 4 (See [3])**.** *Every BL-algebra can be embedded into a poset product of a family of MV-chains and product chains indexed by a forest.*

Hence it is natural to wonder if a BL-algebra is isomorphic to a poset product. Although finite BL-algebras are indeed isomorphic to a poset product of MV-chains (details in [6]), in general the answer is negative (even for BL-chains, as shown in [2]).

Our work is framed in the study of BL-algebras that are isomorphic to a poset product of BL-chains. The aim of this talk is to examine some features of this construction and consider the restriction referred above. Then we will introduce the notions of *indecomposable* and *representable* BL-algebra in the sense of poset product. Our main result provides sufficient conditions for BL-algebras so that they admit a representation as a poset product. The proof, which is constructive, will be outlined.

Theorem 5. *Let* $\mathbf{A}$ *be a BL-algebra. If its subset* $\mathrm{J}(\mathbf{A})$ *of idempotent and join irreducible elements is a well partial order (with the inherited order) such that each* $i \in \mathrm{J}(\mathbf{A})$ *induces a prime filter in* $\mathbf{A}$*, and every* $a \in A$ *has a maximum idempotent element below it, then* $\mathbf{A}$ *is isomorphic to the poset product of a collection of indecomposable BL-chains.*

References

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