Boolean-like Algebras of Dimension n

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We aim at bridging several different areas of logic, algebra and computation — the algebraic analysis of conditional statements in programming languages, the theory of factorizations of algebras, the theory of Boolean vector spaces, the theory of skew Boolean algebras, the investigation into generalizations of classical logic — most of which somehow revolve around the main concept that lies at the crossroads of the three disciplines: the notion of *Boolean algebra* (BA).

There is a thriving literature on abstract treatments of the fundamental if-then-else construct of computer science. The approach followed in this paper originates with Dicker's axiomatisation [3] of Boolean algebras. In [3] the if-then-else was treated as a proper algebraic ternary operation q on an algebra $\mathbf{A}$ with 0 and 1, having the property that for every $a, b \in A$, $q(1, a, b) = b$ and $q(0, a, b) = a$. Such algebras, called Church algebras in [7], are investigated in [8], and will be termed here *algebras of dimension* 2. Apart from Boolean algebras, many algebras investigated in classical mathematics, like rings with unit or ortholattices, have dimension 2.

At the root of all the most important results in the theory of Boolean algebras (Stone's representation theorem included) there is the simple remark that every element $e \neq 0, 1$ of a Boolean algebra $\mathbf{B}$ decomposes $\mathbf{B}$ as a Cartesian product $[0, e] \times [e, 1]$ of *two* nontrivial Boolean algebras. In the more general context of algebras of dimension 2, we say that an element e is 2-*central* if e decomposes the algebra as the Cartesian product of two other algebras univocally determined by e. Algebras of dimension 2 where every element is 2-central, as in the Boolean algebra case, were called *Boolean-like algebras* in [8], since the variety of all such algebras in the type $(q, 0, 1)$ is term-equivalent to the variety of Boolean algebras.

We generalize the previous approach to algebras $\mathbf{A}$ having n designated elements $\mathbf{e}_1, \dots, \mathbf{e}_n$ ($n \geq 2$) and an operation q of arity $n + 1$ (a sort of "generalized if-then-else") satisfying $q(\mathbf{e}_i, x_1, \dots, x_n) = x_i$. The operator q induces, through the so-called n-central elements, a decomposition of $\mathbf{A}$ into n, rather than just 2, factors. These algebras will be called, naturally enough, *algebras of dimension n*, while algebras $\mathbf{A}$ all of whose elements are n-central are given the name of *Boolean-like algebras of dimension n* (nBAs). Examples of algebras having dimension greater than 2 include among others the free algebras, combinatory algebras and Boolean vector spaces of dimension n. Boolean-like algebras of dimension $n > 2$ include p-rings (with p a prime number), n-valued Post algebras, and the algebra of n-central elements in an algebra of dimension n.

Given an algebraic type τ, we prove that nBAs of type τ form a variety, which happens to share many remarkable properties with the variety of Boolean algebras. In particular we show that the variety of all nBAs of type τ is generated by the nBAs of cardinality n. In the pure case (i.e., when the type τ includes just the generalized if-then-else and the n constants), the variety is generated by one n-element algebra $\mathbf{n}$ of the generalized truth values $\mathbf{e}_1, \dots, \mathbf{e}_n$, so that any pure nBA is, up to isomorphism, a subalgebra of $\mathbf{n}^I$ for a suitable set I. One of the most remarkable properties of the 2-element Boolean algebra is the definability of all finite Boolean functions in terms of the connectives AND, OR, NOT. This property is inherited by the algebra $\mathbf{n}$: all finite functions on the n truth values are term-definable, so that the variety of pure nBAs is primal. More generally, every primal variety of type τ is a variety of nBAs.

Skew Boolean algebras (see [2, 6]) constitute a one-pointed noncommutative version of Boolean algebras. We show that skew Boolean algebras and nBA are strongly related in that every pure nBA contains as reducts n isomorphic skew Boolean algebras. More precisely, given a pure nBA $\mathbf{A}$, we prove that, for every $1 \leq i \leq n$, there exists a reduct of $\mathbf{A}$, called SBAi-reduct, which is a right-handed skew Boolean algebra with bottom $\mathbf{e}_i$ and maximal elements $\mathbf{e}_j$, for every $j \neq i$. Moreover, a representation theorem holds: every right-handed skew Boolean algebra can be embedded into the SBAi-reduct of a suitable nBA.

For any pair of distinct constants $\mathbf{e}_i$ and $\mathbf{e}_j$ of an nBA $\mathbf{A}$, we show that $\mathbf{A}$ contains a Boolean algebra $\mathbf{B}_{ij}$ with bottom $\mathbf{e}_i$ and top $\mathbf{e}_j$. This Boolean algebra is used to show that $\mathbf{A}$ is isomorphic to the algebra of n-central elements of the Boolean vector space $(\mathbf{B}_{ij})^n$ (see [4] for basic facts on Boolean vector spaces). Due to this representation theorem, we can view the Boolean vector space generated by the nBA of the n-central elements of a given algebra $\mathbf{A}$ of dimension n as a kind of *linear approximation* of $\mathbf{A}$. Another notable consequence of our result is that Foster's Theorem for primal algebras (cfr. [1, Thm. 7.4]), according to which any member of a variety generated by a primal algebra is a Boolean power of the generator, can be obtained as a corollary.

Just like Boolean algebras are the algebraic counterpart of classical logic CL, for every $n \geq 2$ we define a logic nCL whose algebraic counterpart are nBAs. We show the complete symmetry of the truth values $\mathbf{e}_1, \dots, \mathbf{e}_n$, supporting the idea that nCL is the right generalization of classical logic from dimension 2 to dimension n. Then the universality of nCL is obtained by conservatively translating any n-valued tabular logic (e.g. Post, Łukasiewicz, Tarski and Gödel logics) into it. We define a terminating and confluent term rewriting system to test the validity of formulas of nCL by rewriting. By the universality of nCL, in order to check whether a formula ϕ is valid in a n-valued tabular logic it is enough to see whether the normal form of the translation ϕ^* is the designated value of nCL. Our approach generalizes Zantema and van de Pol work on rewriting of binary decision diagrams [10] and Salibra et al. work on rewriting terms of factor varieties with decomposition operators [9].

References

[1] S. N. Burris, H. P. Sankappanavar, *A Course in Universal Algebra*, Springer, Berlin, (1981).

[2] W. H. Cornish, Boolean skew algebras, *Acta Math. Acad. Sci. Hungar.* **36** (1980) 281–291.

[3] R. M. Dicker, A set of independent axioms for Boolean algebra. *Proc. London Math. Soc.* **3** (1963) 20–30.

[4] S. Gudder, F. Latrémolière, Boolean inner-product spaces and Boolean matrices, *Linear Algebra and its Applications* **431** (2009) 274–296.

[5] Gumm, H.P., Ursini, A., "Ideals in universal algebra", *Alg. Universalis*, 19, 1984, pp. 45–54.

[6] J. Leech, Skew Boolean algebras. *Algebra Universalis* **27** (1990) 497?506.

[7] G. Manzonetto, A. Salibra, Applying universal algebra to lambda calculus. *Journal of Logic and Computation* **20** (2010) 877–915.

[8] A. Salibra, A. Ledda, F. Paoli, T. Kowalski, Boolean-like algebras. *Algebra Universalis* **69** (2013) 113–138.

[9] A. Salibra, G. Manzonetto, G. Favro, Factor Varieties and Symbolic Computation, in: Thirty-First Annual ACM/IEEE Symposium on Logic in Computer Science (2016) 739–748.

[10] H. Zantema, J. van de Pol, A rewriting approach to binary decision diagrams, *The Journal of Logic and Algebraic Programming* **49** (2001) 61–86.