Does First-Order Gödel Logic interpolate?

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1. Introduction

One of the most prominent unknown cases regarding interpolation properties is the infinitely valued first-order Gödel logic $G_{[0,1]}$ on $[0,1]$ (see e.g. [5] for a definition). We present fragments of $G_{[0,1]}$ where interpolation can be shown to hold and approaches to the full case (in case interpolation holds for $G_{[0,1]}$ in general).

2. Solvable Fragments

The weak quantifier fragment: We assume that all quantifiers in $A \to B$ are weak ($\forall$ negative, $\exists$ positive). We construct Herbrand expansions from cut-free proofs in a suitable hypersequent calculus HG for $G_{[0,1]}$. Proofs can be transformed by eliminating weak quantifier inferences: all formulas A_i corresponding to an $\exists$ introduction are selected and the introduction is eliminated using $\bigvee_i A_i$, $\forall$ introductions are eliminated using $\bigwedge_i A_i$. We suppress the inference of weak quantifiers and combine the disjunctions and conjunctions to accommodate contractions. Propositional Gödel logic interpolates, we obtain propositional interpolants for the Herbrand expansions and reintroduce weak quantifiers. Functions in I not in the common language can be eliminated [4].
The prenex case: We assume that A, B in $A \to B$ are prenex and show that $A \to B$ admits Skolemization using the mid-hypersequent-theorem for HG. Substitution of Skolem functions for strong quantifiers is possible, elimination of Skolem functions is similar to the elimination in the proof of the second ε-theorem [7]. We then find a propositional interpolant and eliminate functions not in the common language simultaneously in antecedent/interpolant, interpolant/succedent by reintroducting quantifiers [4]. This corresponds to the general methodology: propositional interpolation and the existence of suitable Skolemizations and Herbrand expansions suffice for first-order interpolation.

3. Approaches to demonstrate the general case

Unusual Skolemizations for $G_{[0,1]}$: Suitable Skolemizations are sufficient for proving interpolation for $G_{[0,1]}$ (Section 1). Candidates are combinations of Skolemizations using relations as in Gödel's completeness proof and Skolemizations closer to semantical notions as in [3].
Prenexation of formulas in $G_{[0,1]}$ using propositional quantifiers: We extend $G_{[0,1]}$ by propositional quantifiers $\nu_I(\forall X A(X)) = \inf(\nu_I(A(d))|\ d \in [0,1])$ and $\nu_I(\exists X A(X)) = \sup(\nu_I(A(d))|d \in [0,1])$. Using the shifts $(\forall u A(u) \to B) \leftrightarrow \forall X \exists u(A(u) \to X \vee X \to B)$, $(\forall X A(X) \to B) \leftrightarrow \forall X' \exists X(A(X) \to X' \vee X' \to B)$,

$(A \to \exists u B(u)) \leftrightarrow \forall X \exists u (A \to X \vee X \to B(u))$, $(A \to \exists X B(X)) \leftrightarrow \forall X' \exists X (A \to X' \vee X' \to B(X))$ prenexation of all formulas is obtained. Other classically valid quantifier shifts are valid for first-order and propositional quantifiers. If we were able to obtain some sort of Herbrand expansions we could proceed as in Sec.2.

Proving a Maehara-style lemma for HG: Note that an obvious Maehara-style lemma to construct interpolants from cut-free proofs exists for the sequent-of-relation calculus for propositional Gödel logic [1]. It is however difficult to extend such a calculus to first-order settings.

The semantic approach: This approach intends to construct countervaluations for $A \to B$ from countervaluations for $A \to I$ or $I \to B$ for all potential I, analogous to the construction in [6] for classical logic. This seems to be the only promising approach to show interpolation for witnessed $G_{[0,1]}$ if interpolation holds there.

Note that the counter example of [8] for constant domain intuitionistic logic, $A \supset B$ with

$$A : \forall x \exists y (P(y) \wedge (Q(y) \supset R(x))) \wedge \neg \forall x R(x)$$

$$B : \forall x (P(x) \supset Q(x) \vee S) \supset S$$

is not a counter example for first-order Gödel logic. The interpolant is

$$I = \neg \forall x \neg\neg Q(x) \wedge \forall x \exists y (P(y) \wedge (Q(y) \to (Q(x) \vee \neg Q(x))) \wedge \neg \forall x (Q(x) \vee \neg Q(x)))$$

$$\vee$$

$$\neg\neg \forall x \neg\neg Q(x) \wedge \forall x \exists y (P(y) \wedge (Q(y) \to Q(x))) \wedge \neg \forall x Q(x).$$

(Another interpolant has been presented in [2].)

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