A LOGICAL VIEW OVER THE VALUED CONSTRAINT SATISFACTION PROBLEM*

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The *constraint satisfaction problem* (CSP, for short) is a well-established framework for the uniform study of a wide range of both theoretical and applied problems, concerning the formulation of constraints and the search of solutions for them. When formulated abstractly and studied in purely logical terms, it is strongly related to model theory and universal algebra.

Formally, a first-order sentence is *primitive positive* (pp, for short) if it is built up from atomic formulas with equality using only conjunction and existential quantifier. The CSP of a finite relational structure $\mathtt{B}$ asks to determine the pp-sentences valid in $\mathtt{B}$, in symbols

$$\mathrm{CSP}(\mathtt{B}) \coloneqq \{\varphi \mid \varphi \text{ is pp-sentence and } \mathtt{B} \vDash \varphi\}.$$

The CSP problem of a finite relational structure is clearly decidable and, more precisely, belongs to the complexity class NP. The most outstanding problem in the field, known as the *Dichotomy Conjecture* (proven in Spring 2017 independently by Zhuk [20] and by Bulatov [3]), shows that for every finite relational structure $\mathtt{B}$, either CSP($\mathtt{B}$) is NP-complete or it is tractable, i.e., solvable in polynomial time [9]. The algebraic approach to CSP has proven very fruitful in this direction and in the general understanding of CSP, leading to the discovery of striking connections of tractability of CSP and the satisfaction of certain Maltsev conditions (see e.g. [5, 4, 12, 1, 2, 17]).

In this work, we will focus on a weighted generalization of the classical CSP, known as *valued constraint satisfaction problem* (VCSP, for short) see eg. [16, 7, 21]. Roughly speaking, weighted structures are generalizations of classical structures in which relations are allowed to take degrees of truth (or, equivalently, payoffs) into a suitable valuation structure (that allows a natural generalization of the notion of pp-formula). Accordingly, the VCSP problem of a finite *weighted* relational structure $\mathtt{B}$ asks to determine an *optimal* solution for any pp-sentence φ, that is a tuple $\vec{b} \in B$ such that for any other tuple $\vec{c} \in B$ the value of φ on $\vec{b}$ is better or equal than its value on $\vec{c}$. As in the classical case, a major challenge is to classify finite weighted structures according to whether their VCSP is tractable or NP-hard.

Our contribution to the VCSP topic begins by observing that VCSP structures can be seen as relational structures of MTL-logics [8]. By doing so, classical constructions and methods developed for the study of CSP can be generalized to the valued case in a way more uniform that previous approaches to VCSP. In particular, the significant majority of works [21, 6, 18, 15, 13] are confined to the algebraic approach for VCSP formulated over a particular valuation structure, namely the set of positive rationals (with or without infinity) equipped with addition. The formulation on more abstract logical terms is amenable to provide a uniform treatment for the VCSP formulated over arbitrary valuation structures.

In the present communication we will elaborate on this interpretation, and the results concerning complexity of VCSP that can be obtained relying in this logical formalization.

Besides the introduction of the VCSP framework based on MTL logics, we will also present a general study on the complexity classification for VCSP over locally finite algebras. In particular, let us remark that in the classical CSP, the fruitful algebraic approach to the study of the

*Based on the longer work [11].

dichotomy is based on Geiger's result [10] characterizing the pp-definable relations of a finite structure `B` by means of *polymorphisms*. We will prove a generalization of Geiger's result to the case of VCSP over locally finite valuation structures, by introducing a new notion of polymorphism for weighted structures which generalizes the classical one. Moreover, in contrast to the previous works on the topic that face the generalization of Geiger's result to valued structures (eg. [6, 14, 21, 15]), we do not modify the language in this process. For this reason our work is not only a logical contribution to the computational study of VCSP (in which expansions preserving tractability are indistinguishable), but also to the development of weighted model theory (where different languages determine different structures).

Further we will show that the VCSP over locally finite valuation structures can be reduced to finitely many CSP's. This reduction to the classical setting has two major consequences. On the one hand, it implies that several *tractability criteria* related to the existence of certain polymorphisms [19] CSP to VCSP. On the other and, this reduction yields that the dichotomy holds for our studied VCSP, given that it holds for classical CSP.

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