Uncertainty of voltage profile in PMU-based Distribution System State Estimation

Carlo Muscas, *Senior Member, IEEE*, Marco Pau, *Student Member, IEEE*,
Paolo Attilio Pegoraro, *Member, IEEE*, Sara Sulis, *Member, IEEE*

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***Abstract*—Distribution System State Estimation (DSSE) tools are required to estimate the real operating conditions of power distribution networks. The accuracy of the quantities estimated through DSSE plays a key role for the effectiveness of the management and controls functions. The paper presents a comprehensive mathematical analysis, aimed at highlighting the most important factors affecting the accuracy of the voltage profile obtained by means of Weighted Least Squares (WLS) estimators. The proposed analysis shows the way in which the uncertainty components of the voltage estimations depend on the measurement system available on the field. In particular, this paper considers synchrophasors measurements provided by Phasor Measurement Units (PMUs) and hybrid measurements systems presenting also conventional devices. Tests performed on a 95-bus network prove the validity of the theoretical analysis and show the importance of the results also in a meter placement perspective, emphasizing the requirements needed to achieve specific accuracy targets.**

***Index Terms*—Distribution System State Estimation, Weighted Least Squares estimators, voltage estimation, uncertainty analysis, meter placement, Phasor Measurement Units**

I. Introduction

The electric distribution systems were designed to transport energy from a limited number of nodes (substations) towards a great amount of consumers. Unidirectional energy flows (from generators to loads) and unidirectional information flows (from users to operators) were expected. However, important changes are taking place. The massive spread of small-size generation plants, Renewable Energy Sources (RES), together with the growing penetration of other Distributed Energy Resources (DERs), like electrical vehicles (EVs) and storage systems as well as new high efficiency residential and commercial appliances are expected [1]. This evolution brings both economic and environmental benefits to the electric system, but at the same time it implies important technical problems and requires significant changes in the distribution networks operation. As a consequence, new management and control functions have to be designed and implemented to guarantee safe operation of the grid [2], [3]. Proper coordination of DERs has also to be provided, so that the greatest technical and economic benefits can be achieved from their deployment [4], [5]. The reliable operation of these functions strictly depends on the awareness about the operating conditions of the grid and on the accuracy of this information [6]. To this purpose, suitable Distribution System State Estimation (DSSE) algorithms have to be used for estimating the operating state of the system. State Estimation (SE) techniques used in transmission grids exploit the redundancy of the measurements collected from the field to filter out the measurement errors and to provide the most likely estimation of the operating conditions of the network [7], [8]. Several methods can be used to perform SE but, generally, the most used algorithms are based on the Weighted Least Squares (WLS) approach.

In distribution systems, the design of accurate DSSE algorithms is one of the most challenging tasks, due to the features of these grids. In particular, a crucial aspect is the high number of nodes with respect to the installed measurement devices. The scarcity of measurement instruments is such that they are very often limited only to the main substation. This problem is generally faced in the estimation process by using the so-called pseudo-measurements, namely information obtained by statistical or historical data about the power injection and/or generation at the nodes. The use of pseudo-measurements makes the DSSE problem solvable, but, since this information is highly uncertain, specific approaches should be adopted to obtain the estimated quantities with sufficient accuracy.

In the literature, the SE accuracy is investigated in several works, focusing on different aspects. In [9], a study is presented aimed at finding the most convenient way to include current phasor measurements into the SE model. In [10], a comprehensive analysis of the measurement uncertainties, considering both meters and instrument transformers, is performed, highlighting how to properly choose the weights into WLS estimators. These papers refer to transmission systems, but most of the reported considerations may also apply to distribution grids. As for the distribution systems, the problem of the accuracy obtainable by means of estimation techniques is still open, due to the heterogeneousness of both measures and measurement systems. In the last years, the authors proposed several papers aimed at providing indications and models for enhancing the accuracy of the estimation process [11]–[13]. However, it is commonly recognized that a significant enhancement of the DSSE accuracy can be achieved only through a suitable upgrade of the available measurement system. Due to obvious economic reasons, it is unrealistic that a high number of measurement devices will be used in the

C. Muscas, P. A. Pegoraro, S. Sulis are with the Department of Electrical and Electronic Engineering of the University of Cagliari, Piazza d'Armi, 09123 Cagliari, Italy (email: [carlo, paolo.pegoraro, sara.sulis]@diee.unica.it). M. Pau in now with the Institute for Automation of Complex Power Systems, Aachen, Germany. email: mpau@eonerc.rwth-aachen.de

near future. Therefore, number, location and accuracy of the measurements to be placed have to be carefully selected on the basis of the accuracy performance required by the management and control functions.

Several papers deal with the issue of meter placement in distribution systems [14]–[18]. Usually, the proposed solutions aim at finding the minimum number of measurement devices to be installed in order to achieve desired accuracy targets for the estimation of specific electrical quantities. Few papers (e.g. [19], [20]) provide some general rules, following gained experience. Recently, some papers (for instance [21],[22]) carried out numerical investigations on the impact of PMUs in the context of DSSE, with different approaches. However, a clear idea on how number, location and features of the used measurements affect the accuracy of the DSSE results is in general missing.

A first step to address this issue was performed in [23], where a mathematical analysis on the main factors affecting the uncertainty of the voltage magnitude estimation in case of conventional measurements was presented. Note that "conventional" here means "non-synchronized", thus without information on the absolute phase-angle. Two main contributions to the overall uncertainty were highlighted. The first one is approximatively constant for all the nodes of the network and only depends on the number and the accuracy of the available voltage measurements. The second term, instead, is variable for each bus and is associated to the estimation uncertainty of the branch currents of the grid.

In this paper, the analysis presented in [23] is reconsidered from scratch in order to allow the introduction of the synchrophasor measurements provided by Phasor Measurement Units (PMUs). The impact of the PMUs on the uncertainty of the results will be evaluated and the accuracy in the estimations of the bus voltage phase-angles will be also discussed. Moreover, an additional study will be performed to show the impact of PMU measurements in hybrid measurement systems, i.e. where both conventional and synchronized measurements are present. Found results provide important information to address the meter placement problem when fixed accuracy targets for the voltage magnitude and phase-angle estimations must be satisfied.

In Section II the considered model of the WLS algorithm used as reference for the theoretical analysis is shortly described. In Section III, the mathematical analysis concerning the use of synchronized measurements for the evaluation of both voltage magnitude and phase-angle estimation uncertainty is discussed. In Section IV, results obtained by means of simulations performed on a 95-bus network are analyzed. Final considerations and conclusions are finally presented.

II. Branch Current Distribution System State Estimation

In the literature, two main categories of WLS algorithms have been conceived for DSSE: node-voltage and branch-current estimators ([11], [20], [24]–[26]). The main difference among the alternative approaches is the choice of the state variables to be used within the algorithm. In [13], it has been demonstrated that, when the same settings are used, WLS algorithms basically provide the same accuracy performance regardless of the choice the state variables. For this reason, in this paper, the branch-current estimator proposed in [11] (indicated in the following as BC-DSSE) is used as reference for the analysis, but found results and reported considerations have general validity for all the WLS estimators.

The general measurement model used for DSSE is:

$$\mathbf{z} = \mathbf{h}(\mathbf{x}) + \mathbf{e} \tag{1}$$

where: $\mathbf{z} = [z_1 \dots z_M]^{\mathrm{T}}$ is the vector of the M measurements gathered from the network and of the chosen pseudo-measurements; $\mathbf{h} = [h_1 \dots h_M]^{\mathrm{T}}$ is the vector of the measurement functions; $\mathbf{x} = [x_1 \dots x_N]^{\mathrm{T}}$ is the vector of the N state variables; $\mathbf{e}$ is the measurement noise vector, usually assumed to be composed by random, zero mean variables, with covariance matrix $\Sigma_{\mathbf{z}}$.

According to [11], in BC-DSSE, the state vector $\mathbf{x}$ has to include a reference bus voltage as well as the rectangular currents in the N_{br} branches. If synchrophasor measurements are used, the state vector $\mathbf{x}$ can be written as:

$$\mathbf{x} = \left[V_s, \theta_s, i_1^r \dots i_{N_{\mathrm{br}}}^r, i_1^x \dots i_{N_{\mathrm{br}}}^x\right]^{\mathrm{T}} \tag{2}$$

with $N = 2 + 2N_{br}$ elements, where V_s and θ_s are the voltage amplitude and phase-angle of the "root" bus chosen as a reference, while i^r and i^x are the real and the imaginary parts of the currents, respectively. It is important to highlight that the root bus can be arbitrarily chosen because it is only necessary to complete the state in the BC formulation in order to accurately estimate the voltage profile [11]. However, different choices of the root bus do not affect the estimates and their uncertainties. With respect to the case of conventional measurements of [23], the root voltage phase-angle is now included and can be estimated. Furthermore, all the phase-angles can be referred to the absolute reference given by the Coordinated Universal Time (UTC) [27].

In the BC-DSSE approach, both voltages and currents are estimated iteratively by means of alternated WLS step and forward sweep step. In the WLS step, the so called normal equations are solved to update the estimation of the state vector, according to the following:

$$\Delta\mathbf{x}_n = \mathbf{x}_{n+1} - \mathbf{x}_n = \mathbf{G}_n^{-1}\mathbf{H}_n^{\mathrm{T}}\mathbf{W}[\mathbf{z} - \mathbf{h}(\mathbf{x}_n)] \tag{3}$$

where: $\mathbf{x_n}$ is the state vector at iteration n; $\mathbf{H}_n$ is the Jacobian of the measurement functions with respect to the state variables; $\mathbf{W}$ is the weighting matrix; $\mathbf{G}_n = \mathbf{H}_n^{\mathrm{T}}\mathbf{W}\mathbf{H}_n$ is the so-called Gain matrix. Coherently with the known measurement properties, for an efficient WLS estimator, $\mathbf{W}$ is given as the inverse of the covariance matrix $\Sigma_{\mathbf{z}}$ of the measurement errors.

At each iteration of the estimation algorithm, a forward sweep calculation follows the WLS step. The forward sweep step computes the network voltages for each node, by a simple evaluation of the voltage drops along the lines, starting from the last estimation of the root bus voltage and the branch currents. The algorithm stops when the update $\Delta\mathbf{x}_n$ of the state vector is smaller than a chosen tolerance.

III. Uncertainty analysis in presence of synchronized measurements

The covariance matrix of the estimated states can be obtained through the inversion of the Gain matrix used in the last iteration of the estimation process. Thus, $\mathbf{G}^{-1}$ is a $N \times N$ matrix having the variances of the estimated states on the diagonal and the covariance terms outside the diagonal. In particular, considering the state vector written as in (2), the element $\mathbf{G}^{-1}(1,1)$ corresponds to the variance $\sigma^2_{\hat{V}_s}$ of the root bus voltage magnitude estimation $\hat{V}_s$, while the element $\mathbf{G}^{-1}(2,2)$ gives the variance $\sigma^2_{\hat{\theta}_s}$ of the estimated phase-angle. It is worth noting that in the following the analysis will be focused on the root bus voltage, but, since the reference bus can be chosen arbitrarily, the results can be generalized to all the nodes of the network.

To analyze the variances of the estimated root bus voltage magnitude and phase-angle, the Gain matrix can be split in four blocks as follows:

$$\mathbf{G} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \tag{4}$$

where $\mathbf{A}$ is a 2×2 matrix, $\mathbf{B}$ is a $2 \times N_{br}$ matrix, $\mathbf{C} = \mathbf{B}^{\mathrm{T}}$ (for the symmetry of the Gain matrix) has $N_{br} \times 2$ size and $\mathbf{D}$ is a $N_{br} \times N_{br}$ squared matrix.

The inverse of this block matrix can be written as:

$$\mathbf{G}^{-1} = \begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{bmatrix} \tag{5}$$

Focusing on the 2×2 covariance matrix of the root bus voltage phasor Σ_s, it is possible to use the Woodbury matrix identity to obtain:

$$\begin{aligned} \mathbf{\Sigma_s} &= \mathbf{G}^{-1}\big|_{m=1,2;n=1,2} = (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} = \\ &= \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} \end{aligned} \tag{6}$$

The second block in the main diagonal of (5) is the covariance matrix $\mathbf{\Sigma}_I$ of the rectangular currents estimations, thus (6) can be expressed as follows:

$$\mathbf{\Sigma_s} = \begin{bmatrix} \sigma^2_{\hat{V}_s} & \sigma_{\hat{V}_s,\hat{\theta}_s} \\ \sigma_{\hat{V}_s,\hat{\theta}_s} & \sigma^2_{\hat{\theta}_s} \end{bmatrix} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\mathbf{\Sigma}_I\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1} \tag{7}$$

where $\sigma_{\hat{V}_s,\hat{\theta}_s}$ is the covariance between magnitude and phase-angle of the root bus voltage estimation. (7) is the generalization to synchronized measurements of the expression found in [23] for conventional ones.

A. Analysis of contribution of measurements to the Gain matrix

To understand which terms are involved in the uncertainty expression of the voltage estimations, it is necessary to analyze the contributions, pertaining to different measurement types, forming the Gain matrix. Three types of measurements can be distinguished: voltage magnitude measurements, voltage phase-angles, and other measurements (powers or currents). The Gain matrix can be analyzed by distinguishing the contributions coming from the voltage synchrophasor measurements from the remaining ones. It is worth noting that, in the BC-DSSE formulation, all the power measurements are converted in equivalent current measurements (see [25] for details). This leads to a slight approximation in the Gain matrix, since the effect of the power measurements on the voltage state is lost. However, the approximation exists only in the Gain matrix but not in the BC-DSSE results, since the equivalent measurements are refined at each iteration considering the last estimation of the voltage profile. The Gain matrix can be written as:

$$\begin{aligned} \mathbf{G} = \mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H} &= \begin{bmatrix} \mathbf{H}_V^{\mathrm{T}} & \mathbf{H}_\theta^{\mathrm{T}} & \mathbf{H}_I^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{W}_V & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_\theta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{W}_I \end{bmatrix} \begin{bmatrix} \mathbf{H}_V \\ \mathbf{H}_\theta \\ \mathbf{H}_I \end{bmatrix} \\ &= \mathbf{H}_V^{\mathrm{T}}\mathbf{W}_V\mathbf{H}_V + \mathbf{H}_\theta^{\mathrm{T}}\mathbf{W}_\theta\mathbf{H}_\theta + \mathbf{H}_I^{\mathrm{T}}\mathbf{W}_I\mathbf{H}_I = \mathbf{G}_V + \mathbf{G}_\theta + \mathbf{G}_I \end{aligned} \tag{8}$$

where the subscripts V, θ and I indicate voltage magnitudes, voltage phase-angles and other (power and/or currents) measurements. Correspondingly, $\mathbf{G}_V$, $\mathbf{G}_\theta$ and $\mathbf{G}_I$ are the contributions to the Gain matrix, $\mathbf{W}_V$, $\mathbf{W}_\theta$ and $\mathbf{W}_I$ are the weighting sub-matrices and $\mathbf{H}_V$, $\mathbf{H}_\theta$ and $\mathbf{H}_I$ are the associated Jacobians.

Maintaining the same terminology, it is possible to split the contributions of (4). Since the current measurements can be directly expressed in terms of the corresponding currents state variable (see [11] for the details) and thus do not have, in the Jacobian, any derivative term with respect to the root bus voltage, $\mathbf{A}_I$, $\mathbf{B}_I$ and $\mathbf{C}_I$ are null matrices and it is thus possible to show that:

$$\mathbf{G} = \begin{bmatrix} \mathbf{A}_V + \mathbf{A}_\theta & \mathbf{B}_V + \mathbf{B}_\theta \\ \mathbf{B}_V^{\mathrm{T}} + \mathbf{B}_\theta^{\mathrm{T}} & \mathbf{D}_V + \mathbf{D}_\theta + \mathbf{D}_I \end{bmatrix} \tag{9}$$

Thus, the computation of the sub-matrices $\mathbf{A}$ and $\mathbf{B}$ appearing in (7) has been reduced to the analysis of $\mathbf{H}_V^{\mathrm{T}}\mathbf{W}_V\mathbf{H}_V + \mathbf{H}_\theta^{\mathrm{T}}\mathbf{W}_\theta\mathbf{H}_\theta$.

Considering the derivative terms appearing in the Jacobians $\mathbf{H}_V$ and $\mathbf{H}_\theta$, as detailed in the Appendix, the following expression for $\mathbf{A}$ can be found:

$$\mathbf{A} = \sum_{i \in \Lambda_i} \begin{bmatrix} w_{V_i}\cos^2\theta_{is} + w_{\theta_i}\frac{\sin^2\theta_{is}}{V_i^2} & V_s\left(w_{V_i} - \frac{w_{\theta_i}}{V_i}\right)\frac{\sin 2\theta_{is}}{2} \\ V_s\left(w_{V_i} - \frac{w_{\theta_i}}{V_i}\right)\frac{\sin 2\theta_{is}}{2} & V_s^2\left(w_{V_i}\sin^2(\theta_{is}) + w_{\theta_i}\frac{\cos^2\theta_{is}}{V_i^2}\right) \end{bmatrix} \tag{10}$$

where: i is the index of the node where the voltage phasor measurement is placed; θ_{is} is the phase-angle difference between node i and the root bus; w_{V_i} and w_{θ_i} are, respectively, the weights associated to the voltage amplitude and phase-angle measurement in node i.

The above passages and considerations hold, independently of the network topology, for both transmission and distribution systems. However, as shown in the following, the typical characteristics of distribution systems allow specific approximations that give a better insight into uncertainty sources. This aspect is of paramount importance in distribution, where the scarcity of measurement devices asks for a careful choice of measurement types and locations.

B. Causes of uncertainty for the estimation of the root bus voltage magnitude and phase-angle

Taking into account that distribution systems usually have short lines with low impedances, the phase-angles θ_{is} are generally small and they can be, in first approximation, considered equal to zero. With this assumption, the mutual influences of voltage amplitude and phase-angle are decoupled and, from (10), the following expression holds for the voltage amplitude uncertainty:

$$\sigma^2_{\hat{V}_s} = \sigma^2_a + \sigma^2_b \simeq \frac{1}{\sum_i w_{V_i}} + \frac{1}{(\sum_i w_{V_i})^2}[\mathbf{b}_1^T \mathbf{\Sigma}_I \mathbf{b}_1] \quad (11)$$

where $\mathbf{b}_1$ is the transpose of the first row of $\mathbf{B}$.

From (11), it is possible to observe that, like for the conventional voltage measurements (see [23]), the two terms influencing the overall uncertainty of $\hat{V}_s$ have a different source. The first term, σ^2_a, is a component only dependent on the number of PMU voltage amplitude measurements and their accuracy. If, for the sake of simplicity, voltage measurements with the same standard deviation $\sigma_{V_{\text{PMU}}}$ are taken into account, the following holds:

$$\sigma^2_a \simeq \frac{1}{M_{\text{PMU}} w_{V_{\text{PMU}}}} = \frac{\sigma^2_{V_{\text{PMU}}}}{M_{\text{PMU}}} \quad (12)$$

where M_{PMU} is the total number of voltage phasor measurements available on the network and $w_{V_{\text{PMU}}}$ is the weight of all the voltage measurements.

As for the second term in (11), the elements of $\mathbf{b}_1$, with the aforementioned assumptions are:

$$\mathbf{b}_1(j) \simeq \begin{cases} \sum_i \lambda_{ji} R_{ji} w_{V_i} & \text{if } j \leq N_{br} \\ \sum_i \lambda_{ji} X_{ji} w_{V_i} & \text{if } j > N_{br} \end{cases} \quad (13)$$

where: i is the index of the node where the voltage measurement is placed; λ_{ji} is a logic value equal to 1 if the branch j is in the path considered in the Jacobian between node i and the root bus and 0 otherwise; $R_{ji} = (-r_j \cos\theta_i - x_j \sin\theta_i)$ and $X_{ji} = (x_j \cos\theta_i - r_j \sin\theta_i)$ are the derivatives of the voltage magnitude measurement in node i with respect to the real and imaginary part of the current in branch j, respectively, with r_j and x_j representing the resistance and reactance of branch j. σ^2_b thus results in a long sum of elements, strictly related to the uncertainty of the voltage drops between each measured node and the chosen root bus. As a consequence, changing the root bus impacts on $\mathbf{b}_1$ and then on the uncertainty term σ^2_b, reflecting the different levels of uncertainty achievable on different nodes. It is important to recall that changing the root bus does not affects the estimates and the uncertainties of the voltage profile, thus such approach allows exploiting the presented expressions to better analyse where and how uncertainty arises at each node.

As for the voltage phase-angle uncertainty the following expression can be derived:

$$\sigma^2_{\hat{\theta}_s} = \sigma^2_c + \sigma^2_d \simeq \frac{1}{\sum_{i\in\Lambda_i} \frac{V_s^2}{V_i^2} w_{\theta_i}} + \frac{1}{(\sum_{i\in\Lambda_i} \frac{V_s^2}{V_i^2} w_{\theta_i})^2}[\mathbf{b}_2^T \mathbf{\Sigma}_I \mathbf{b}_2] \quad (14)$$

where $\mathbf{b}_2$ is the transpose of the second row of $\mathbf{B}$. Analogously to the case of the voltage amplitude estimation, two terms influence the overall uncertainty of the estimated phase-angle $\hat{\theta}_s$. The first term, σ^2_c, only depends on the number of voltage phase-angle measurements, that is on the number of PMUs, and on their accuracy and represents a lower bound for the uncertainty of the phase-angles. If voltage phase-angle measurements with the same uncertainty $\sigma_{\theta_{\text{PMU}}}$ are considered and the realistic assumption that, in practice, $V_s \simeq V_i \simeq 1$ p.u. holds, (14) can be simplified as follows:

$$\sigma^2_c \simeq \frac{1}{\sum_{i\in\Lambda_i} \frac{1}{\sigma^2_{\theta_i}}} \simeq \frac{\sigma^2_{\theta_{\text{PMU}}}}{M_{\text{PMU}}} \quad (15)$$

Analogously to the voltage amplitude profile, the voltage phase-angle profile also depends on the term σ^2_d that includes the additional uncertainty term related to the quality in the knowledge of the current flows along the network.

In the Section IV, the relevance of such terms will be illustrated by means of simulation examples with different measurement configuration systems. As a final consideration it is useful to recall that, besides the aforementioned contributions, the uncertainty in the knowledge of the network parameters also affects the overall WLS DSSE uncertainty, impacting on its formulation as discussed in [28]. However, a thorough analysis of such contribution is outside the scope of this paper, which is focused on the impact of the measurement system uncertainties.

C. Hybrid measurement system

In real scenarios, hybrid measurement systems, with the availability of both conventional and synchronized measurements, can be present on the field. Even in this case, the contribution brought by the different types of measurements to the final voltage estimation uncertainty can be investigated through the decomposition of the Gain matrix. Indicating with the subscript "PMU" the matrices related to voltage phasor measurements coming from PMUs, with V_C the matrices related to voltage magnitude measurements provided by conventional meters, respectively, and again with the subscript "I" the terms related to all the other measurements (powers and currents), it is possible to write:

$$\mathbf{G} = \begin{bmatrix} \mathbf{H}^{\text{T}}_{\text{PMU}} & \mathbf{H}^{\text{T}}_{V_C} & \mathbf{H}^{\text{T}}_I \end{bmatrix} \begin{bmatrix} \mathbf{W}_{\text{PMU}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{V_C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{W}_I \end{bmatrix} \begin{bmatrix} \mathbf{H}_{\text{PMU}} \\ \mathbf{H}_{V_C} \\ \mathbf{H}_I \end{bmatrix} = \\ = \mathbf{G}_{\text{PMU}} + \mathbf{G}_{V_C} + \mathbf{G}_I \quad (16)$$

Dividing the Gain matrix in the same four blocks as in (4):

$$\mathbf{G} = \begin{bmatrix} \mathbf{A}_{\text{PMU}} + \mathbf{A}_{V_C} & \mathbf{B}_{\text{PMU}} + \mathbf{B}_{V_C} \\ \mathbf{B}^{\text{T}}_{\text{PMU}} + \mathbf{B}^{\text{T}}_{V_C} & \mathbf{D}_{\text{PMU}} + \mathbf{D}_{V_C} + \mathbf{D}_I \end{bmatrix} \quad (17)$$

Focusing on $\mathbf{A}$, containing the information about the theoretical limit of uncertainty for the voltage estimation, and considering the phase-angle differences approximatively equal to 0, the analysis of the Gain matrix elements brings to:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{V_{\text{PMU}}} + \mathbf{A}_{V_C} & 0 \\ 0 & \mathbf{A}_{\theta_{\text{PMU}}} \end{bmatrix} \quad (18)$$

where $A_{V_{\text{PMU}}}$ and $A_{\theta_{\text{PMU}}}$ are the contributions brought by the PMUs measurements.

Equation (18) shows that, in a hybrid measurement system, the theoretical limit of uncertainty for the voltage phase-angle estimations is affected only by the PMU phase-angle measurements. As a result, the same considerations presented for (14) and (15) still hold. As for the voltage magnitude estimations, instead, the uncertainty limit is affected by the voltage magnitude measurements of both PMUs and conventional meters. The different contribution coming from PMUs and conventional meters can be highlighted analyzing the term $\mathbf{A}(1,1)$. Considering the derivatives appearing in the Jacobian for the voltage magnitude measurements, and with the aforementioned approximation for the phase-angle differences, it is possible to find:

$$\mathbf{A}(1,1) = \sum_{i=1}^{M_{\text{PMU}}} w_{V_{\text{PMU}_i}} + \sum_{i=1}^{M_{V_C}} w_{V_{C_j}} \tag{19}$$

where M_{V_C} is the number of conventional voltage measurements and the $w_{V_{C_j}}$ are the corresponding weights. If, for the sake of simplicity, we suppose to have M_{PMU} voltage PMU measurements with the same uncertainty and, analogously, M_{V_C} conventional voltage measurements with same uncertainty, the following theoretical limit for the voltage magnitude uncertainty can be found:

$$\begin{aligned} \mathbf{A}^{-1}(1,1) &= \frac{1}{M_{\text{PMU}} w_{V_{\text{PMU}}} + M_{V_C} w_{V_C}} = \\ &= \frac{\sigma^2_{V_C}}{M_{V_C} + k M_{\text{PMU}}} \end{aligned} \tag{20}$$

where $\sigma^2_{V_C}$ is the variance of the conventional voltage magnitude measurements, whereas k is defined as:

$$k = \frac{\sigma^2_{V_C}}{\sigma^2_{V_{\text{PMU}}}} \tag{21}$$

Equation (20) shows that, in a hybrid measurement system, the theoretical uncertainty limit for the voltage magnitude estimation can be evaluated as the ratio between conventional voltage measurement uncertainty and total number of voltage measurements. However, the number of PMUs to be considered in (20) has to involve a corrective factor k, which depends on the ratio between the variances of conventional and synchrophasor measurements. Generally, since the PMU accuracies are significantly better than those of the conventional meters, the value of k is larger than 1. The information associated to the coefficient k basically indicates that the placement of each PMU would be equivalent to the installation of k conventional meters. Such information can be important above all in a meter placement perspective, for addressing the choice of the measurements to be used in a possible upgrade of an existing measurement system.

IV. Tests and results

The presented theoretical analysis has been validated by means of tests performed on a 95-bus network[1] showed in Fig.1. Data of the 95-bus network can be found in [29]. The use of this network permits to easily compare the obtained results with previously presented results [14], [23]. The overall structure of this Section is the same as [23], so that a comparison with the results obtained with conventional measurements devices can be easily made.

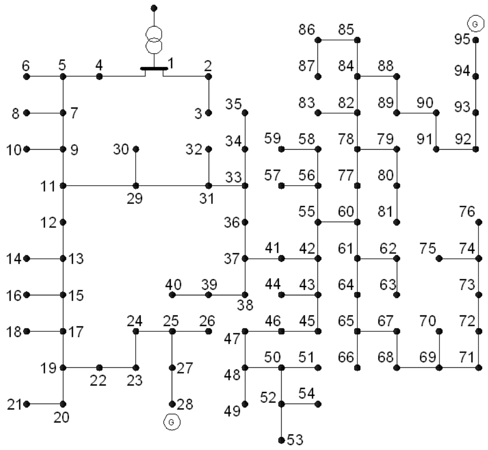

Fig. 1. Test system 95-bus

Several tests have been done considering different measurement systems and assumptions. The reference operating conditions of the network have been calculated by means of a load flow calculation, starting from the nominal values.

Monte Carlo (MC) approach performing 25000 trials for each test series has been used to evaluate the uncertainty of the results. Such number of trials assures the repeatability of the results. Each MC trial simulates a possible operating condition of the network. Pseudo-measurements are assumed for the power injections and/or generation of all the nodes. For this type of data, a Gaussian distribution has been assumed. As usual in this kind of studies, a standard deviation equal to one third of the variation interval (50% of the rated value) has been taken into account.

PMU measurements are also considered as normally distributed. For each trial, the amplitude and phase angle PMU measurements are randomly extracted according to the assumed distributions. It is important to recall that, in the used BC estimator [11], PMU measurements are included in rectangular coordinates and thus the correlation arising in the transformation has to be duly included in $\mathbf{W}$ [2].

It is worth noting that the accuracy of the phasor measurements is often underestimated in the literature. As an example, accuracies equal to $0.02\,\%$ for voltage and current magnitudes and equal to $0.01°$ for phase-angles are assumed, according to data derived from [30]. However, by reading more in detail PMU data-sheets (see for example the accuracy specifications reported in [31]) it is possible to realize that such values are not realistic. In this data-sheet the measurement accuracy is indicated as $0.1\,\%$ Total Vector Error (TVE) maximum, plus estimator error, that is the error of the measurement algorithm.

[1] As for the numeration of the branches, each branch index is given by the node number of its end node (the largest one), decreased by one.

[2] For PMU measurements, due to the small measurement errors the impact of the coordinate transformation slightly affects the normality of the variables (as clear from the first order uncertainty propagation law). However, as aforementioned, in the WLS estimation, the correct covariances are considered.

This is the accuracy actually obtainable for the synchrophasor estimation. For this reason, in this paper, a maximum TVE of 1% (i.e. the limit defined by [32] for most steady state compliance tests) is assumed for most of the test cases, and such value is translated, for simplicity, into $0.7\,\%$ and 0.7 crad (1 crad = 0.01 rad) for voltage magnitude and phase-angle, respectively.

A. Validation of the theoretical analysis of the uncertainty sources

The first test set has been performed to verify the goodness of the presented analysis. Four PMUs measuring voltage synchrophasors have been considered on nodes 1, 11, 28 and 37. This will be referred to as basic measurement system. Fig. 2 shows the expanded uncertainty for the voltage magnitude estimations (coverage factor equal to 3 and percentage evaluated with respect to the base voltage of the network) obtained using both (11) and the MC simulations. Since (11) gives the uncertainty of the root bus, to complete the theoretical uncertainty profile for all the nodes and validate the expression, (11) has been applied for each bus, by moving the root bus and recomputing. A similar approach has been adopted also for the phase angle uncertainties in the following.

Eq. (11) is composed by two terms, σ_a^2 and σ_b^2. σ_a^2 is a component depending only on accuracy and number of the available voltage measurements. Therefore, it is a constant term of uncertainty, common to all the nodes, and is the lowest value of uncertainty achievable with a given measurement system. Such limit is plotted in Fig. 2 as “Theoretical limit” with a black dash-dot line. In this case, it is 0.35% evaluated from (11). Actually, it should be reminded that the Gain matrix used in the mathematical analysis was slightly approximated because of the conversion of power measurements in equivalent currents. For this reason, there are some cases where this limit could be slightly exceeded. However, in general (and in most of the cases of practical interest for the distribution systems), σ_a^2 can be assumed as reference for the maximum accuracy achievable through a given measurement system. σ_b^2 is instead an additional term of uncertainty depending on the voltage drop uncertainty.

Differences between ”Theoretical” and ”Monte Carlo” uncertainties are basically negligible. Given the considered measurement system, the lowest uncertainties can be observed around node 11. The area covering this node is sufficiently close to all the installed measurements and hence the voltage drops of the path toward the measured nodes are limited. For this reason, the uncertainty values are similar to the theoretical limit. It also possible to observe the uncertainty corresponding to the voltage magnitude of the node 95. This node is far from the measured nodes, furthermore, it is equipped with a generator providing a large current. For this reason, it presents the most significant uncertainty value. The same analysis has been performed even for the phase-angle measurements, and analogous considerations still hold, as shown in Fig. 3.

Fig. 4 shows the expanded uncertainty obtainable for the voltage magnitude starting from different measurement systems. In particular, in the figure the hybrid system is composed by conventional voltage measurements in the nodes 1 and 28 (accuracy equal to $1\,\%$), while PMUs voltage measurements are considered for the nodes 11 and 37. It is possible to observe as all the trends follow the theoretical approach and that, obviously, the hybrid system obtain accuracies included among those obtainable with the PMUs and those available with the conventional measures.

Given the basic PMU measurement system, Fig. 5 shows the variations of uncertainty for different load conditions. In particular, scenarios with the power injections scaled at 125%, 100%, 75% and 50% are presented. Lower values for

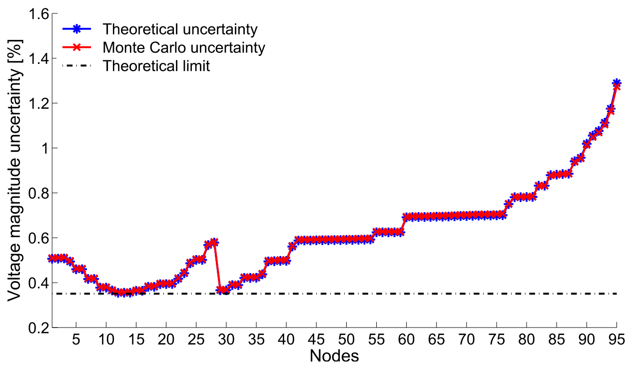

Fig. 2. Expanded uncertainty of the voltage magnitude estimations: theoretical approach and Monte Carlo simulation

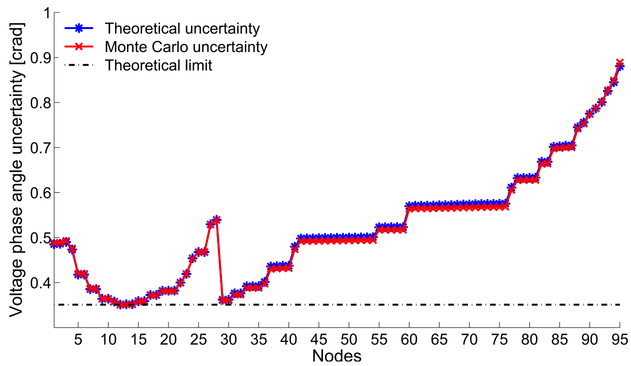

Fig. 3. Expanded uncertainty of the voltage phase-angle estimations: theoretical approach and Monte Carlo simulation

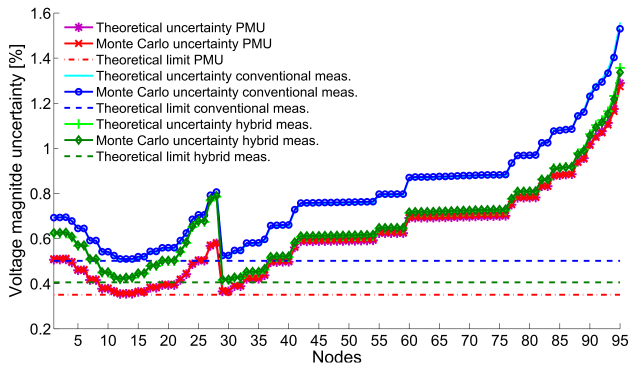

Fig. 4. Expanded uncertainty of the voltage magnitude estimations: theoretical approaches, Monte Carlo simulations, and accuracy limit for different measurement systems

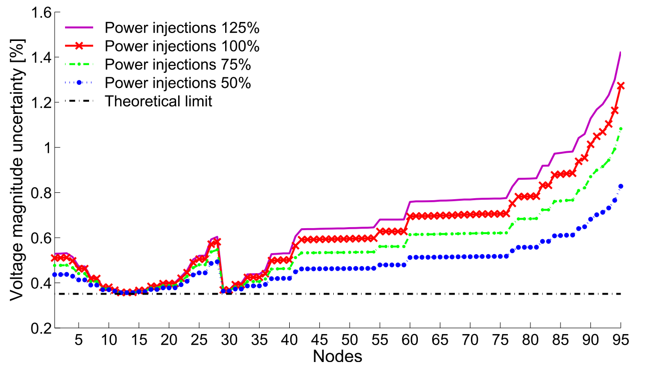

Fig. 5. Expanded uncertainty of voltage magnitude estimations with different load conditions

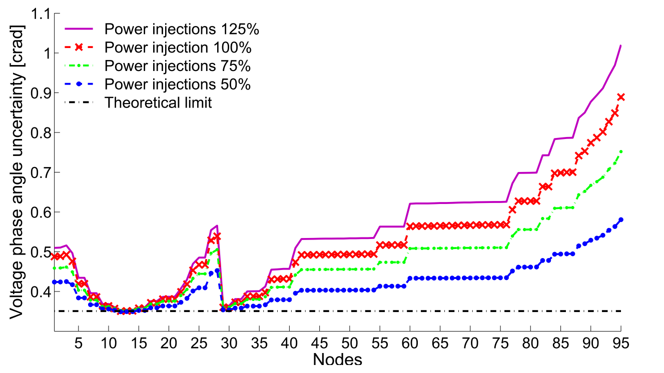

Fig. 6. Expanded uncertainty of voltage phase-angle estimations with different load conditions

powers drawn by the loads imply lower absolute values for the uncertainties of the currents and thus a smaller impact on the overall uncertainty of the magnitude voltage estimations. Fig. 6 shows the same kind of results for the voltage phase-angles estimations. Even in this case, it is possible to observe the role played by the voltage drops. The uncertainty in the estimation of the voltage phase-angle can reach 1 crad in the nodes far from measurement devices.

The impact of the assumption made for the pseudo-measurement values on the estimation results has been also considered. Fig. 7 highlights the effect of improving the knowledge of the behaviour of loads and generators. Relying on more accurate pseudo-measurements means improving also the knowledge of the branch currents and this is reflected in the results by a smaller impact of the voltage drop uncertainties. In this case, an hypothetical uncertainty of the pseudo-measurements lower than 10% would lead to an overall uncertainty of the voltage estimation very close to the theoretical limit. The same kind of results were obtained with conventional measurements in [23].

B. Theoretical analysis of the uncertainty sources in a meter placement perspective

To improve the accuracy of the results a measurement system with additional measurements should be taken into account. More voltage phasor measurements reduce the theoretical limit of the voltage uncertainty. Additional current

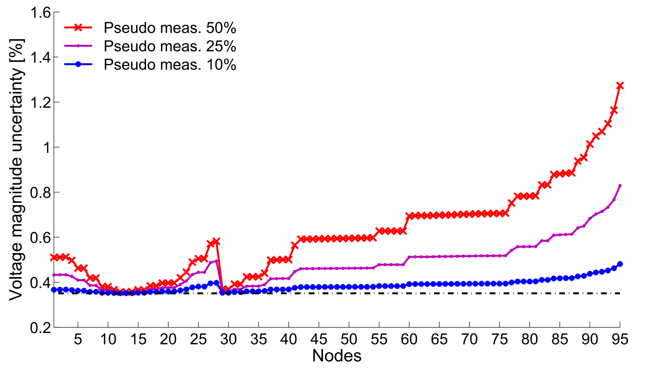

Fig. 7. Expanded uncertainty of voltage magnitude estimations in case of pseudo-measurements with different accuracies

phasor measurements, instead, enhance the accuracy estimation of some highly uncertain currents. Different solutions can be adopted to find this more "robust" measurement system depending on type, accuracy and location of the considered devices. Optimal or sub-optimal meter placement techniques have been recently proposed in literature. However, some of them address the issue in a simplified way (see for example [14], [15]): the main idea is adding, iteratively, devices on the nodes with the largest uncertainty. However, such solution can be often not convenient.

Several tests have been done to address this aspect. In particular, starting from the basic measurement system and pseudo-measurements with accuracy equal to 50%, two different measurement systems have been considered. In the so called "Solution A", six additional PMUs measuring phasor voltage (in nodes 4, 25, 27, 93, 94 and 95) have been placed according to the previous simplified approach. As alternative configuration,"Solution B" is composed with additional PMUs measuring the voltage phasors in nodes 27 and 95 and the current phasors of the branches 26 and 94, plus an added PMU channel measuring the current of branch 3. Such configuration has been chosen on the basis of the analytical results previously found: voltage measurements have been added to reduce the first term of uncertainty below a desired threshold (0.3% for amplitudes), and current measurements have been placed in branches carrying large currents to reduce the second component of uncertainty. Since the exact position of the voltage measurements is marginally important, they are placed in the nodes adjacent to the current measurements in order to exploit only one PMU measuring both voltage and current.

The results obtained with such solutions are reported in Fig. 8. It can be observed that Solution A brings a general benefit for many nodes, with respect to the results reported in Fig. 2, but some significantly high uncertainties in the estimations still exist. With Solution B, instead, the difference in the uncertainty of the estimation results is minimum and an accuracy target, for example, equal to 0.33% can be achieved for all the nodes even with a lower number of measurement devices. Therefore, as it can be observed in Fig. 8, adding voltage measurements allows a reduction of the theoretical limit, but some peaks of uncertainty can still remain if a suitable placement of flow measurements (power

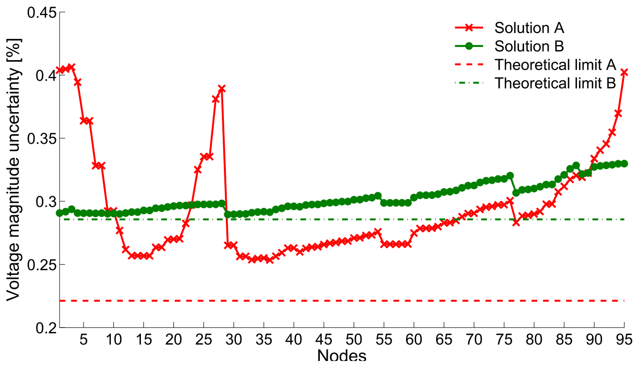

Fig. 8. Expanded uncertainty of voltage magnitude estimations with different measurement configurations

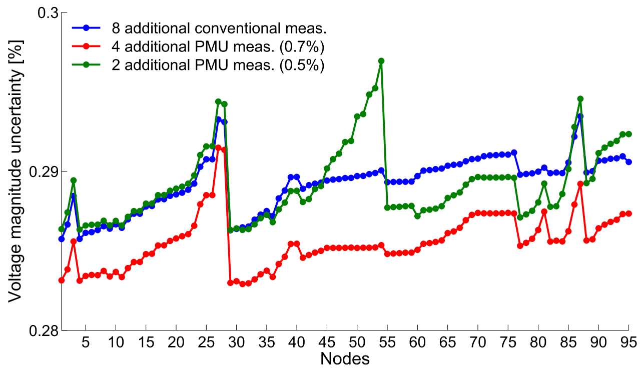

Fig. 9. Expanded uncertainty of voltage magnitude estimations with different measurement systems

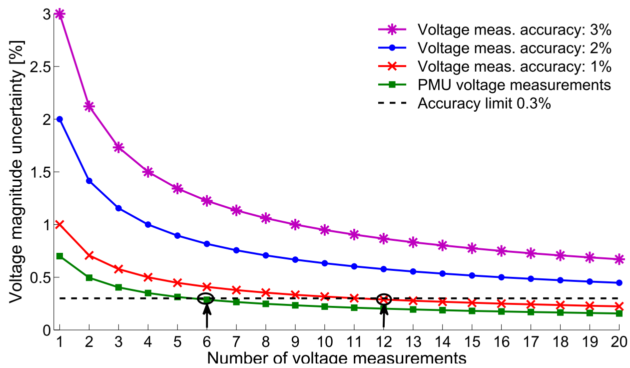

Fig. 10. Theoretical limit of the expanded uncertainty of voltage estimations with increasing number of voltage measurements

or current) is not provided. The proposed analysis, providing clear indications on composition of the uncertainty of the results, can thus be useful in a meter placement perspective.

Another test is proposed to show the usefulness of the relationships found in Section III-C in case of hybrid measurement systems. A starting configuration with four traditional measurement points has been considered, having voltage magnitude measurements at nodes 1, 11, 28 and 37 and power measurements at branches 3, 12, 27 and 40. The considered accuracies for voltage magnitude and power measurements are 1% and 3%, respectively. Starting from this configuration, an upgrade of the measurement system has been looked for to achieve voltage magnitude estimation accuracies lower than 0.3%. Two alternative options have been analyzed. The first one is the installation of additional traditional meters providing measurements of the bus voltage magnitude and of one power in an adjacent branch. The second one is the deployment of PMUs providing measurements for the bus voltage and one branch current. Fig. 9 shows the obtained results. When conventional measurements are considered, the number of devices needed to obtain the desired accuracy target is quite large and leads to the installation of 8 additional meters. If PMUs with accuracy of 0.7% for the amplitude measurements are taken into account, instead, it is possible to observe that the coefficient k of equation (21) is approximatively equal to 2 and, thus, only 4 additional measurement points are required. Moreover, Fig. 9 shows the results obtained even by considering a PMU with magnitude measurement accuracies equal to 0.5%. In this case, the coefficient k in (21) is equal to 4 (each PMU is equivalent to 4 conventional meters) and thus only two additional measurement points would be needed to achieve the desired performance.

C. Benefits for the estimation accuracy as a function of the number of measurements

It is worth noting that, in general, the marginal benefits arising from the installation of an additional voltage measurement decrease for increasing number of such measurements. In Fig. 10 the value of the theoretical limit of uncertainty for an increasing number of devices is reported, for several measurement accuracies and different measurement devices. As clearly proven by (12), it is possible to observe that all these trends are proportional to $1/\sqrt{M_V}$. A similar trend was empirically found, considering the TVE metric, for instance in [21]. The number of measurements providing estimation results respecting a given accuracy limit (in this case assumed, as an example, equal to 0.3%) is highlighted in case of PMU and accurate conventional voltage measurements. This limit is respected in case of both six PMUs on the field with accuracy equal to 0.7%, and twelve conventional voltage measurements with accuracy equal to 1%. The advantage of having PMUs on the field is evident.

The obtained trends can provide a first reference for the uncertainty of the voltage profile to be expected depending on the number and accuracy of available voltage measurements.

V. Conclusions

In this paper, a mathematical study on the components of uncertainty affecting the voltage profile provided by WLS estimators in a distribution system context has been presented. The analysis has been performed focusing on the impact brought by PMU measurements and evaluating the main sources of uncertainty for both magnitudes and phase-angles of the voltage estimations. Obtained results show that both the electrical quantities have two main sources of uncertainty, one of which allows evaluating the maximum accuracy achievable through the available measurement system. The impact brought by the PMUs in a hybrid measurement system has been also studied, showing the effects deriving from the different levels of accuracy between conventional and synchrophasor measurements. Performed tests prove the validity of the developed analysis, point out the effects brought by the different terms of uncertainty on the final estimation outcomes, and emphasize the importance of the found results in a meter placement perspective.

Appendix

To obtain the 2×2 portion $\mathbf{A}$ of the Gain matrix related to the root phasor, the contributions of the voltage and phase-angle measurements, given by each PMU to the matrix elements referring to the root voltage state variables (magnitude V_s and phase-angle θ_s), have to be computed. Thus, considering the generic PMU on node i, that measures V_i and θ_i, and using the same notation of the Section III, the following relationship holds:

$$\mathbf{A} = \sum_{i \in \Lambda_i} \mathbf{A}_{V_i} + \mathbf{A}_{\theta_i} \tag{A.1}$$

where Λ_i is the set of the n_{PMU} indices of the nodes monitored by PMUs while the matrices $\mathbf{A}_{V_i}$ and $\mathbf{A}_{\theta_i}$ are expressed as follows:

$$\begin{aligned} \mathbf{A}_{V_i} &= w_{V_i}\ \mathbf{h}_{V_i}^{\mathrm{T}} \mathbf{h}_{V_i}\big|_{m=1,2;n=1,2} \\ \mathbf{A}_{\theta_i} &= w_{\theta_i}\ \mathbf{h}_{\theta_i}^{\mathrm{T}} \mathbf{h}_{\theta_i}\big|_{m=1,2;n=1,2} \end{aligned} \tag{A.2}$$

where $\mathbf{h}_{V_i}$ and $\mathbf{h}_{\theta_i}$ are the rows of the Jacobian corresponding to the voltage magnitude and phase measurements, respectively, and w_{V_i} and w_{θ_i} are the corresponding weights. From (A.2), it is clear that $\mathbf{A}_{V_i}$ and $\mathbf{A}_{\theta_i}$ are the submatrices related to the root node variables of the contribution to the Gain matrix of the ith voltage measurement and of the corresponding phase-angle measurement. For this reason, to find $\mathbf{A}$, it is sufficient to compute (A.2) for each voltage phasor measurement, by means of the products of the first two components of $\mathbf{h}_{V_i}$ and $\mathbf{h}_{\theta_i}$, that is of the derivatives of the measurement functions with respect to V_s and θ_s. Since $\boldsymbol{v}_i$ is expressed as follows:

$$\begin{aligned} \boldsymbol{v}_i &= \boldsymbol{v}_s - \sum_{k \in \Gamma_i} \boldsymbol{z}_k \boldsymbol{i}_k = \\ &= V_s \cos\theta_s + jV_s \sin\theta_s - \sum_{k \in \Gamma_i} (r_k + jx_k)(i_k^r + j i_k^x) = \\ &= V_s \cos\theta_s + jV_s \sin\theta_s - \sum_{k \in \Gamma_i} (r_k i_k^r - x_k i_k^x) + \\ &- j \sum_{k \in \Gamma_i} (r_k i_k^x + x_k i_k^r) \end{aligned} \tag{A.3}$$

where Γ_i is the set of branches connecting the node i to the root bus, $\boldsymbol{z}_k = (r_k + jx_k)$ is the line impedance of branch k and $\boldsymbol{i}_k$ its branch current, the voltage magnitude V_i results as follows:

$$\begin{aligned} V_i &= \boldsymbol{v}_i e^{-j\theta_i} = \mathrm{Re}\left[\boldsymbol{v}_i e^{-j\theta_i}\right] = V_s \cos(\theta_s - \theta_i) + \\ &- \cos\theta_i \sum_{k \in \Gamma_i} (r_k i_k^r - x_k i_k^x) - \sin\theta_i \sum_{k \in \Gamma_i} (r_k i_k^x + x_k i_k^r) \end{aligned} \tag{A.4}$$

The derivatives of V_i result as follows:

$$\frac{\partial V_i}{\partial V_s} = \cos(\theta_i - \theta_s) \ \text{ and } \ \frac{\partial V_i}{\partial \theta_s} = V_s \sin(\theta_i - \theta_s) \tag{A.5}$$

The derivatives of θ_i require more passages to obtain useful expressions. In fact, the phase-angle at node i can be expressed as:

$$\theta_i = \arctan(v_i^x / v_i^r) = \frac{V_s \sin\theta_s - \sum_{k \in \Gamma_i} (r_k i_k^x + x_k i_k^r)}{V_s \cos\theta_s - \sum_{k \in \Gamma_i} (r_k i_k^r - x_k i_k^x)} \tag{A.6}$$

and, after few passages, its derivatives with respect to V_s and θ_s are:

$$\begin{aligned} \frac{\partial \theta_i}{\partial V_s} &= \frac{-\sin\theta_s \sum_{k \in \Gamma_i} (r_k i_k^r - x_k i_k^x)}{V_i^2} + \\ &+ \frac{\cos\theta_s \sum_{k \in \Gamma_i} (r_k i_k^x + x_k i_k^r)}{V_i^2} \end{aligned} \tag{A.7}$$

$$\begin{aligned} \frac{\partial \theta_i}{\partial \theta_s} &= V_s \left[\frac{V_s - \cos\theta_s \sum_{k \in \Gamma_i} (r_k i_k^r - x_k i_k^x)}{V_i^2} + \right. \\ &\left. - \frac{\sin\theta_s \sum_{k \in \Gamma_i} (r_k i_k^x + x_k i_k^r)}{V_i^2} \right] \end{aligned} \tag{A.8}$$

Considering that, in a dual manner with respect to (A.4), V_s can be written as a function of $\boldsymbol{v}_i$, as follows:

$$\begin{aligned} V_s &= \boldsymbol{v}_s e^{-j\theta_s} = \mathrm{Re}\left[\boldsymbol{v}_s e^{-j\theta_s}\right] = V_i \cos(\theta_i - \theta_s) + \\ &+ \cos\theta_s \sum_{k \in \Gamma_i} (r_k i_k^r - x_k i_k^x) + \sin\theta_s \sum_{k \in \Gamma_i} (r_k i_k^x + x_k i_k^r) \end{aligned} \tag{A.9}$$

and the following identity holds:

$$\begin{aligned} \mathrm{Im}\left[\boldsymbol{v}_s e^{-j\theta_s}\right] &= V_i \sin(\theta_i - \theta_s) - \sin\theta_s \sum_{k \in \Gamma_i} (r_k i_k^r - x_k i_k^x) \\ &+ \cos\theta_i \sum_{k \in \Gamma_i} (r_k i_k^x + x_k i_k^r) = 0 \end{aligned} \tag{A.10}$$

the derivatives in (A.7) and (A.8) can be simplified as follows:

$$\frac{\partial \theta_i}{\partial V_s} = -\frac{\sin(\theta_i - \theta_s)}{V_i} \ \text{ and } \ \frac{\partial \theta_i}{\partial \theta_s} = \frac{V_s \cos(\theta_i - \theta_s)}{V_i} \tag{A.11}$$

Given (A.5) and (A.11), it is possible to compute the terms of (A.2). The following matrices are then obtained:

$$\begin{aligned} \mathbf{A}_{V_i} &= w_{V_i} \begin{bmatrix} \left(\frac{\partial V_i}{\partial V_s}\right)^2 & \frac{\partial V_i}{\partial V_s}\frac{\partial V_i}{\partial \theta_s} \\ \frac{\partial V_i}{\partial \theta_s}\frac{\partial V_i}{\partial V_s} & \left(\frac{\partial V_i}{\partial \theta_s}\right)^2 \end{bmatrix} = \\ &= \frac{1}{\sigma_{V_i}^2} \begin{bmatrix} \cos^2(\theta_i - \theta_s) & V_s \sin(\theta_i - \theta_s)\cos(\theta_i - \theta_s) \\ V_s \sin(\theta_i - \theta_s)\cos(\theta_s - \theta_i) & V_s^2 \sin^2(\theta_i - \theta_s) \end{bmatrix} \end{aligned} \tag{A.12}$$

$$\begin{aligned} \mathbf{A}_{\theta_i} &= w_{\theta_i} \begin{bmatrix} \left(\frac{\partial \theta_i}{\partial V_s}\right)^2 & \frac{\partial \theta_i}{\partial \theta_s}\frac{\partial \theta_s}{\partial V_i} \\ \frac{\partial \theta_i}{\partial \theta_s}\frac{\partial \theta_i}{\partial V_s} & \left(\frac{\partial \theta_i}{\partial \theta_s}\right)^2 \end{bmatrix} = \\ &= \frac{1}{\sigma_{\theta_i}^2} \begin{bmatrix} \frac{\sin^2(\theta_i - \theta_s)}{V_i^2} & -\frac{V_s}{V_i}\sin(\theta_i - \theta_s)\cos(\theta_i - \theta_s) \\ -\frac{V_s}{V_i}\sin(\theta_i - \theta_s)\cos(\theta_i - \theta_s) & \frac{V_s^2 \cos^2(\theta_i - \theta_s)}{V_i^2} \end{bmatrix} \end{aligned} \tag{A.13}$$

and thus, using (A.12) and (A.13) in (A.1) and defining $\theta_{is} = \theta_i - \theta_s$, $\mathbf{A}$ can be expressed as follows:

$$\mathbf{A} = \sum_{i\in\Lambda_i} \begin{bmatrix} \frac{\cos^2\theta_{is}}{\sigma^2_{V_i}} + \frac{\sin^2\theta_{is}}{\sigma^2_{\theta_i} V_i^2} & V_s\left(\frac{1}{\sigma^2_{V_i}} - \frac{1}{\sigma^2_{\theta_i} V_i}\right)\frac{\sin 2\theta_{is}}{2} \\ V_s\left(\frac{1}{\sigma^2_{V_i}} - \frac{1}{V_i\sigma^2_{\theta_i}}\right)\frac{\sin 2\theta_{is}}{2} & \frac{V_s^2 \sin^2(\theta_{is})}{\sigma^2_{V_i}} + \frac{V_s^2\cos^2\theta_{is}}{\sigma^2_{\theta_i} V_i^2} \end{bmatrix} \tag{A.14}$$

The inverse of $\mathbf{A}$ can be easily obtained from (A.1) and, in particular, its diagonal elements that correspond to the terms of interest for voltage and phase-angle uncertainty evaluation. Since, in practical conditions of distribution systems, $\theta_{is} \simeq 0$, $V_i \simeq 1\,\text{p.u.}$ and $V_s \simeq 1\,\text{p.u.}$, such elements become:

$$\mathbf{A}^{-1}(1,1) \simeq \frac{1}{\sum_{i\in\Lambda_i}\frac{\cos^2\theta_{is}}{\sigma^2_{V_i}}} \simeq \frac{1}{\sum_{i\in\Lambda_i}\frac{1}{\sigma^2_{V_i}}} \simeq \frac{\sigma^2_{V_{\text{PMU}}}}{n_{\text{PMU}}} \tag{A.15}$$

$$\mathbf{A}^{-1}(2,2) \simeq \frac{1}{\sum_{i\in\Lambda_i}\frac{V_s^2\cos^2\theta_{is}}{\sigma^2_{\theta_i} V_i^2}} \simeq \frac{1}{\sum_{i\in\Lambda_i}\frac{1}{\sigma^2_{\theta_i}}} \simeq \frac{\sigma^2_{\theta_{\text{PMU}}}}{n_{\text{PMU}}} \tag{A.16}$$

where the last term of (A.15) is obtained assuming the same standard uncertainty for the PMU voltage amplitude measurements at each node and the last term of (A.16) is obtained assuming that all the PMUs have the same phase-angle measurement accuracy.

Acknowledgements

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Carlo Muscas (M'98) received the M.S. (cum laude) degree in electrical engineering from the University of Cagliari, Cagliari, Italy, in 1994.

He was Assistant Professor with the University of Cagliari from 1996 to 2001. He has been an Associate Professor of Electrical and Electronic Measurement with the University of Cagliari, since 2001, where he is currently the Chairman of the Council for the M. S. degree in Electrical Engineering.

His current research interests include the electrical and electronic measurements, with special attention to the study of power quality phenomena, to the measurement of synchronized phasors, and to the implementation of distributed measurement systems for modern electric grid. He has authored and co-authored more than 120 scientific papers.

Mr. Muscas is currently an Associate Editor of the IEEE TRANSACTIONS ON INSTRUMENTATION AND MEASUREMENT.

Marco Pau (S'14-M'15) received the M.S. degree in electrical engineering and the Ph.D. degree in electronic engineering and computer science from the University of Cagliari, Italy, in 2011 and 2015, respectively.

Currently, he is a research associate at the Institute for Automation of Complex Power Systems at the E.On Energy Research Center, RWTH Aachen University.

His main research activity concerns measurement systems applied to electric distribution grids and designed for Distribution System State Estimation.

Paolo Attilio Pegoraro (S'03-M'06) received the M.S. (cum laude) degree in telecommunications engineering and the Ph.D. degree in electronic and telecommunications engineering from the University of Padua, Padua, Italy, in 2001 and 2005, respectively.

He currently holds a post-doctoral position with the University of Cagliari, Cagliari, Italy, where he focuses on the development of new measurement techniques for modern power networks.

His current research interests include the measurement of synchronized phasors and the state estimation for electric distribution grids.

Sara Sulis (S'04, M'06) received the Ph.D. Degree in Industrial Engineering from the University of Cagliari in 2006, where she is currently Assistant Professor of Electrical and Electronic Measurements. During 2013, she obtained the Faculty Course Development Award, funded by the Instrumentation and Measurement Society and the Habilitation to the position of Associate Professor.

Dr. Sulis is co-author of more than sixty scientific papers. She is reviewer for international journals and for international conferences. She is a member of IEEE Instrumentation and Measurement Society and of the GMEE, Associazione Italiana Gruppo Misure Elettriche ed Elettroniche.

Her current research activity mostly concerns distributed measurement systems designed to study reliable methodologies to perform both state estimation and harmonic sources estimation of distribution networks.