RESEARCH MEETING ON NON-LOCAL OPERATORS

Cagliari, October, 6-8, 2016

Università degli Studi di Cagliari

Weierstraß-Institut für Angewandte Analysis und Stochastik

PRIN Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations

ERC Grant EPSILON

Contents

1 Introduction and acknowledgments

We are delighted to be hosting the RESEARCH MEETING ON NON-LOCAL OPERATORS (RMNLO) here at the University of Cagliari (Italy). The aim of such an activity is to gather European researchers, all specialist on non-local operators and related subjects, and let them share their results and discuss further developments and open problems.

Mainly, this meeting was possible thanks to the generous availability of all the speakers who participated in the activity. Moreover, we also appreciate the effort of all the other colleagues who, directly or indirectly, offered their useful contribution in the organization of the RMNLO. Finally, we would like to express our gratitude to the *Weierstrass Institute for Applied Analysis and Stochastics, Berlin (WIAS)*, the *European Research Council: project EPSILON* and the *PRIN Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations*; without the support of these sponsors the conference would have been not possible.

Cagliari, 6 October 2016

The Scientific Committee

Antonio Iannizzotto (*Univ. Cagliari*)
Enrico Valdinoci (*Univ. Melbourne, Univ. Milano* and *WIAS*)
Giuseppe Viglialoro (*Univ. Cagliari*)

2 INVITED SPEAKERS

1. BEGOÑA BARRIOS BARRERA (UNIV. LA LAGUNA TENERIFE)
2. LORENZO BRASCO (UNIV. FERRARA)
3. ELEONORA CINTI (UNIV. TORINO)
4. GAMZE DÜZGÜN (HACETTEPE UNIV. ANKARA)
5. ANTONIO GRECO (UNIV. CAGLIARI)
6. GABRIELE GRILLO (POLITECNICO MILANO)
7. TOMMASO LEONORI (UNIV. GRANADA)
8. SUNRA MOSCONI (UNIV. CATANIA)
9. DIMITRI MUGNAI (UNIV. PERUGIA)
10. ROBERTA MUSINA (UNIV. UDINE)
11. GIAMPIERO PALATUCCI (UNIV. PARMA)
12. BENEDETTA PELLACCI (UNIV. NAPOLI "PARTHENOPE")
13. IRENEO PERAL ALONSO (UNIV. AUTONOMA MADRID)
14. PATRIZIA PUCCI (UNIV. PERUGIA)
15. MARCO SQUASSINA (UNIV. CATTOLICA DEL SACRO CUORE)

3 Abstracts of the contributions

Monotonicity of solutions for some nonlocal elliptic problems in half-spaces

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Abstract

Along this talk we will consider classical solutions of the semilinear fractional problem

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

where $(-\Delta)^s$, $0 < s < 1$, stands for the fractional laplacian, $N \geq 2$, $\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^N : x_N > 0\}$ is the half-space and $f \in C^1$ is a given function. With no additional restriction on the function f, we show that bounded, nonnegative, nontrivial classical solutions are indeed positive in $\mathbb{R}^N_+$ and verify

$$\frac{\partial u}{\partial x_N} > 0 \quad \text{in } \mathbb{R}^N_+.$$

This is in contrast with previously known results for the local case $s = 1$, where nonnegative solutions which are not positive do exist and the monotonicity property above is not known to hold in general even for positive solutions when $f(0) < 0$ (see for instance [1, 2, 3]).

This work is joint with L. Del Pezzo (UBA, Argentina), J. García-Melián (ULL) and A. Quaas (Universidad Técnica Federico Santa María, Chile).

References

[1] H. Berestycki, L. Caffarelli, L. Nirenberg, *Further qualitative properties for elliptic equations in unbounded domains*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), 69–94.

[2] C. Cortázar, M. Elgueta, J. García-Melián, *Nonnegative solutions of semilinear elliptic equations in half-spaces*, J. Math. Pures Appl. (2016), in press.

[3] A. Farina, B. Sciunzi, *Qualitative properties and classification of nonnegative solutions to* $-\Delta u = f(u)$ *in unbounded domains when* $f(0) < 0$, Rev. Mat. Iberoam. (2016), in press.

Regularity issues for some nonlocal and nonlinear elliptic equations

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Abstract

In this talk, I will review some regularity results for weak solutions of nonlocal variants of the $p-$Laplace equation. The model case is given by the Euler-Lagrange equation of an Aronszajn–Gagliardo–Slobodeckij seminorm. In particular, I will present a higher differentiability result for solutions, recently obtained in collaboration with Erik Lindgren (KTH).

Quantitative flatness results and BV estimates for nonlocal minimal surfaces

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Abstract

We establish quantitative flatness results in low dimensions and universal BV estimates for minimizers of a wide class of nonlocal interactions that generalize fractional perimeters. In the particular case of the fractional perimeter (which was introduced in [1]) it is known that minimizers in the all $\mathbb{R}^2$ are necessarily halfplanes. We give a quantitative version of this result, in the following sense: we prove that minimizers in a large ball B_R are close (in the L^1 and stronger senses) to be a halfplane in B_1— with a quantitative estimate depending on R. Moreover, we also obtain BV estimates and sharp energy estimates in every dimension for stable sets, which are objects that generalize stable minimal surfaces. As a byproduct of our main results, we also prove that halfplanes are the only minimizers in the whole $\mathbb{R}^2$ of the anisotropic fractional perimeters. Our proofs are based on a non-trivial refinement of a method introduced in [4] to prove flatness of cones minimizing the fractional perimeter. This is a joint work with J. Serra and E. Valdinoci

References

[1] L. Caffarelli, J.-M. Roquejoffre, and O. Savin, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. 63 (2010), 1111–1144.

[2] E. Cinti, J. Serra, and E. Valdinoci, *Quantitative flatness results and BV estimates for stable nonlocal minimal surfaces*, preprint.

[3] O. Savin and E. Valdinoci, *Some monotonicity results for minimizers in the calculus of variations*, J. Funct. Anal. 264 (2013), no. 10, 2469–2496.

[4] O. Savin and E. Valdinoci, *Regularity of nonlocal minimal cones in dimension 2*, Calc. Var. Partial Differential Equations, 48 (2013), no. 1–2, 33–39.

Three Nontrivial Solutions for a Nonlinear Fractional Laplacian Problem

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Abstract

We consider the problem

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is a bounded domain with a C^2 boundary, $s \in (0,1)$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. The fractional Laplacian operator is defined for any sufficiently smooth function $u : \mathbb{R}^N \to \mathbb{R}$ and all $x \in \mathbb{R}$ by

$$(-\Delta)^s u(x) = C_{N,s} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \tag{1.2}$$

where $B_\varepsilon(x)$ is the open ball of radius $\varepsilon > 0$ centered at x and $C_{N,s} > 0$ is a suitable normalization constant. For the existence of three nontrivial solutions of problem (1.1), we make use of the second deformation theorem and some spectral properties of $(-\Delta)^s$ if $f(x,.)$ is sublinear at infinity and make use of the mountain pass theorem and Morse theory if $f(x,.)$ is superlinear at infinity.

Work in collaboration with Antonio Iannizzotto.

References

[1] F.G. Duzgun, A. Iannizzotto, Three nontrivial solutions for nonlinear fractional Laplacian equations, *Adv. Nonlinear Anal.* (to appear)

[2] A. Iannizzotto, S. Mosconi, M. Squassina, H^s versus C^0-weighted minimizers, *Nonlinear Differ. Equ. Appl.* **22** (2015), 477–497.

On symmetry and convexity of solutions of elliptic and parabolic problems

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Abstract

This talk deals with symmetry, as well as convexity of solutions of elliptic and parabolic problems involving the fractional Laplacian. More precisely, I will report on three recent results:

- Radial symmetry for an overdetermined problem related to the equation $(-\Delta)^s u = 1$ (joint work with R. Servadei [1]);
- Characterization (dichotomy) of convex functions over the whole space $\mathbb{R}^N$ satisfying $(-\Delta)^s u = f(u)$ in some open subset $\Omega \subset \mathbb{R}^N$ [2];
- Convexity of solutions of the initial-value problem for the fractional heat equation $u_t + (-\Delta)^s u = 0$ with convex initial data (joint work with A. Iannizzotto [3]).

Proofs are based on maximum and comparison principles, and on the representation formula for the canonical solution of the parabolic initial-value problem.

References

[1] A. Greco and R. Servadei, Hopf's lemma and constrained radial symmetry for the fractional Laplacian, *Math. Res. Lett.* **23** (2016), 863–885

[2] A. Greco, Convex functions over the whole space locally satisfying fractional equations, *Minimax Theory Appl.* **2** (2017), in print

[3] A. Greco and A. Iannizzotto, Sign-changing solutions of the fractional heat equation: existence and convexity, submitted manuscript

Existence and uniqueness of solutions to the porous media equation with measure data in the local and nonlocal setting

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Abstract

We consider a weighted version of the fractional porous media equation, and develop existence and uniqueness results for solutions corresponding to measure data, uniqueness being not immediate even in the unweighted case. We also show how methods of the same type, together with delicate potential theoretic results and geometric methods, allow to treat similar problems also for local problems posed on negatively curved Riemannian manifolds.

References

[1] G. Grillo, M. Muratori, F. Punzo, Fractional porous media equations: existence and uniqueness of weak solutions with measure data, *Calc. Var. Partial Diff. Equ.* **54** (2015), 3303-3335.

[2] G. Grillo, M. Muratori, F. Punzo, The porous medium equation with measure data on negatively curved Riemannian manifolds, preprint 2016.

Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations

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Abstract

Let $\mathscr{L}$ be a nonlocal operator and consider the parabolic problem

$$\begin{cases} u_t + \mathscr{L} u &= f \quad \text{in } Q_T \equiv \Omega \times (0,T), \\ u(x,t) &= 0 \quad \text{in } \left(\mathbb{R}^N \setminus \Omega\right) \times (0,T), \\ u(x,0) &= 0 \quad \text{in } \Omega, \end{cases}$$

where and Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$ with Lipschitz boundary and $T > 0$.

I want to discuss existence, uniqueness and summability results for the solution u of the above problem with respect to the summability of the datum f. The elliptic case is also considered.

References

[1] T. Leonori, Peral, Ana Primo, F. Soria, *Basic estimates for solution of elliptic and parabolic equations for a class of nonlocal operators*, Discrete and Continuous Dynamical Systems, **35** (2015) 6031–6068

Pohozaev identity for nonlinear nonlocal operators

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Abstract

We will discuss a work in progress with L. Brasco and M. Squassina on a Pohozaev identity for entire solutions to

$$(-\Delta_p)^s u = f(u) \qquad \text{in } \mathbb{R}^N, \tag{1}$$

where $(-\Delta_p)^s$ is the fractional p-Laplacian, defined for $p > 1$ and $s \in]0,1[$ as the subdifferential of the convex functional

$$u \mapsto \frac{1}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy.$$

The identity has been proved in [1] under decay assumptions on u when $p = 2$. In the general case $p \neq 2$ only an inequality in the corresponding Pohozaev identity is proved to hold in [2]. The main difference with respect to the classical case is the very low Sobolev regularity of solutions to (1) (especially for small values of s), which forces the use of new techniques to handle the approximation scheme.

We will finally discuss the main issues one faces when treating the problem in star-shaped domains and delicate boundary terms are expected to appear.

References

[1] X. Ros Oton, J. Serra, The Pohozaev idenity for the Fractional Laplacian, *Arch. Rat. Mech. Anal.* **213** (2014), 587–628

[2] X. Ros-Oton, J. Serra, Nonexistence results for nonlocal equations with critical and supercritical nonlinearities, *Comm. Partial Differential Equations* **40** (2015), 115–133.

Existence and multiplicity results for the fractional Laplacian in bounded domains

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Abstract

First, we study existence results for a linearly perturbed elliptic problem driven by the spectral fractional Laplacian. Then, we show a multiplicity result when the perturbation parameter is close to the eigenvalues of the leading operator. This latter result is obtained by exploiting the topological structure of the sublevels of the associated functional.
Joint paper with Dayana Pagliardini (SNS Pisa).

Variational inequalities for fractional Laplacians

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Abstract

We study obstacle problems for the Dirichlet and the Spectral fractional Laplacians of order $s \in (0,1)$. Existence, continuous dependence and regularity results are proved under mild assumptions on the data. The results are taken from [1] and [2].

References

[1] R. Musina, A.I. Nazarov, K. Sreenadh, Variational inequalities for the fractional Laplacian, *Potential Anal.* (in press)

[2] R. Musina, A.I. Nazarov, Variational inequalities for the spectral fractional Laplacian, *Comput. Math. Math. Phys.* (in press)

The obstacle problem for nonlinear integro-differential operators

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Abstract

We deal with *the obstacle problem* for a class of nonlinear nonlocal equations, which include as a particular case some fractional Laplacian-type equations, driven by the following operators defined on fractional Sobolev spaces,

$$\mathscr{L}u(x) = \int_{\mathbb{R}^n} K(x,y)|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\,\mathrm{d}y, \qquad x \in \mathbb{R}^n; \tag{1}$$

for any differentiability order $s \in (0,1)$ and any summability exponent $p > 1$, K is a suitable kernel of order (s,p) with merely measurable coefficients. The integral may be singular at the origin and must be interpreted in the appropriate sense.

The obstacle problem involving fractional powers of the Laplacian operator naturally appears in many contexts, such as in the study of anomalous diffusion, in the quasi-geostrophic flow problem, and in pricing of American options. It can be stated in several ways. Roughly speaking, a solution u to the fractional obstacle problem is a minimal supersolution to the equation $\mathscr{L}u = 0$ above an *obstacle function* h.

In the linear case when $p = 2$ and when the kernel K reduces to the Gagliardo kernel $K(x,y) = |x-y|^{-n-2s}$, a large treatment of the fractional obstacle problem can be found for instance in the fundamental papers by Caffarelli, Figalli, Salsa, and Silvestre; see [4]. In the more general framework considered here, the panorama seems rather incomplete. Clearly, the main difficulty into the treatment of the operators $\mathscr{L}$ in (1) relies in their very definition, which combines the typical issues given by its *nonlocal* feature together with the ones given by its *nonlinear* growth behavior; also, further efforts are needed due to the presence of merely measurable coefficient in the kernel K.

In the present talk, we will show the existence and uniqueness of the solution to the obstacle problem. We will also show that the regularity of the solutions to the obstacle problem inherits the regularity of the obstacle, both in the case of boundedness and (Hölder) continuity ([3]), up to the boundary.

References

[1] A. Di Castro, T. Kuusi, G. Palatucci: Local behavior of fractional p-minimizers. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **33** (2016), 1279–1299

[2] J. Korvenpää, T. Kuusi, G. Palatucci: Fractional superharmonic functions and the Perron method for nonlinear integro-differential equations. *arXiv*:1605.00906.

[3] J. Korvenpää, T. Kuusi, G. Palatucci: The obstacle problem and Hölder regularity up to the boundary for nonlocal integro-differential operators. *Calc. Var. Partial Differential Equations* **55** (2016), No. 3, Art. 63.

[4] S. Salsa: The problems of the obstacle in lower dimension and for the fractional Laplacian. In *Regularity estimates for nonlinear elliptic and parabolic problems*. *Lecture Notes in Math.* **2045** (2012), Springer, Heidelberg, 153–244.

A Logistic equation with nonlocal diffusion: looking for the best choice for survival

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Abstract

Abstract: We will study Neumann boundary value problems under the action of fractional (spectral) diffusion. We will focus on the study of the optimization of the positive principal eigenvalue in dependence on the indefinite potential, on the motility function and on the fractional order. This analysis is important for the optimization of the survival threshold in populations dynamic.

References

[1] B. Pellacci, G.Verzini, Optimization of the positive principal eigenvalue for indefinite fractional Neumann problems. Preprint.

Sharp solvability conditions for a fractional parabolic problem involving Hardy potential*

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Abstract

In this talk we study the influence of the Hardy potential in the fractional heat equation. In particular, we consider the problem

$$(P_\theta) \quad \begin{cases} u_t + (-\Delta)^s u &= \lambda \dfrac{u}{|x|^{2s}} + \theta u^p + cf \text{ in } \Omega \times (0,T), \\ u(x,t) &> 0 \text{ in } \Omega \times (0,T), \\ u(x,t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ u(x,0) &= u_0(x) \text{ if } x \in \Omega, \end{cases}$$

where $N > 2s$, $0 < s < 1$, $(-\Delta)^s$ is the fractional Laplacian of order $2s$, $p > 1$, $c, \lambda > 0$, $\theta = \{0,1\}$, and u_0, $f \geq 0$ are in a suitable class of functions.

The main results are:

1. Optimal summability of the data with respect to the spectral value λ to have positive solution.
2. *Instantaneous and complete blow up* in the linear problem (P_0)
3. The existence of a critical power $p_+(s,\lambda)$ in the semilinear problem (P_1).

References

[1] B. Abdellaoui, M. Medina, I. Peral, A. Primo, *Optimal results for the fractional heat equation involving the Hardy potential.* Nonlinear Analysis 140 (2016), 166-207.

[2] B. Barrios, M. Medina, I. Peral, *Some remarks on the solvability of non local elliptic problems with the Hardy potential.* Commun. Contemp. Math. 16 (2014), no. 4, 1350046, 29 pp. DOI: 10.1142/S0219199713500466

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Fractional Laplacian models for nonlocal elasticity of composite materials

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Abstract

In this talk we present the models contained in [1, 2], which describe the elastic behavior of composite materials and involve the *fractional Laplacian* operator $(-\Delta)^s$. In dimension one we first consider

$$(\mathscr{D}) \qquad \begin{cases} -cu'' + \kappa(-\Delta)^s u + V(x)u = \dfrac{f(x)}{E} & \text{in } (-L,L), \\ u = 0 & \text{in } \mathbb{R}\setminus(-L,L), \end{cases}$$

which represents the case of a nonlocal elastic rod restrained at the ends. Here $c \geq 0$, $\kappa \geq 0$ and $c+\kappa > 0$. The terms $V(x)u$ and $f(x)$ represent external springs. Moreover, the potential V is a non–negative and bounded measurable weight and its stiffness is related to the position of the point along the rod. Finally, E is the Young modulus of the classical elasticity.

We completely solve the problem $(\mathscr{D})$ showing the existence of a unique weak solution and providing natural sufficient conditions under which this solution is actually a classical solution of the problem. For $(\mathscr{D})$ we also perform numerical simulations and a parametric analysis, in order to highlight the response of the rod, in terms of displacements and strains, according to different values of the mechanical characteristics of the material. The main novelty of this approach is the extension of the central difference method by the numerical estimate of the fractional Laplacian operator through a finite–difference quadrature technique.

For $N = 1$ and for higher dimensions $N \geq 2$ we study other general problems for which the existence of weak solutions is proved via variational methods.

The talk is based on the results contained in the papers [1, 2].

References

[1] G. Autuori, F. Cluni, V. Gusella, P. Pucci, Mathematical models for nonlocal elastic composite materials, to appear in *Adv. Nonlinear Anal.*, pages 39.

[2] G. Autuori, F. Cluni, V. Gusella, P. Pucci, Effects of the fractional Laplacian order on the nonlocal elastic rod response, submitted for publication, pages 16.

Bourgain-Brézis-Mironescu limits for magnetic spaces

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Abstract

We discuss a Bourgain-Brézis-Mironescu type formula for a class of nonlocal magnetic spaces, building a bridge with the classical theory of magnetic Sobolev spaces.

4 Program of the meeting

All the invited speakers will present their contributions at the rooms *B* and *D* of the Department of Mathematics and Computer Science (Palazzo delle Scienze, Via Ospedale, 72, Cagliari), and according to the following program.

	6/10	7/10	8/10
Chair/Room		*Pucci/D*	*Grillo/B*
9:00-9:45		Peral	Squassina
9:45-10:30		Barrios	Pellacci
10:30-11:15		Leonori	Mosconi
11:15-11:45		**Coffee break**	**Coffee break**
11:45-12:30		Musina	Brasco
12:30-13:15		Cinti	
Chair/Room	*Peral/B*	*Squassina/B*	
15:00-15:15	**Opening**		
15:15-16:00	Pucci	Grillo	
16:00-16:30	**Coffee break**	**Coffee break**	
16:30-17:15	Mugnai	Greco	
17:15-18:00	Palatucci	Düzgün	