The Decomposition of Linearly Ordered Pseudo-Hoops

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1 Extended Abstract

Pseudo-hoops are partially ordered residuated monoids, not necessarily commutative, with the order given by the left- and right-divisibility, and they are also meet-semilattices, satisfying a divisibility condition. The structure was initially introduced and studied by Bosbach under the name of hoops, and no assumption on the commutativity of the monoid operation was made. However, subsequent studies focused on commutative hoops, until the non-commutative structure was again put in circulation by Georgescu, Leuștean and Preoteasa under the name pseudo-hoops.

Pseudo-hoops are weak structures. By adding axioms to them, one can obtain the algebras of non-commutative fuzzy logic: the pseudo-BL algebras, the pseudo-Wajsberg or the pseudo-MV algebras, the pseudo-product algebras. Pseudo-hoops are the algebraic counterpart of falsehood-free fragments of non-commutative fuzzy logics.

Mostert and Shields proved that every continuous $t$-norm on $[0,1]$ is locally (i.e. on specific open sub-intervals) isomorphic to: either Lukasiewicz’s $t$-norm ($x \circ y = sup(0, x+y-1)$), or to the product $t$-norm ($x \circ y = xy$ product of reals), and, in between different components, the $t$-norm is Gödel ($x \circ y = min\{x, y\}$).

For linear commutative hoops several decompositions as ordinal sums were obtained. Hájek has generalized this result for linear BL-algebras, obtaining a decomposition into Wajsberg and product algebras. Cignoli, Esteva, Godo and Torrens continue the work of Hájek and obtain a decomposition with Gödel, product and Wajsberg components. Using a completely different construction, Agliano and Montagna obtain a decomposition of commutative hoops with only Wajsberg components. The constructions of Hájek and of Agliano and Montagna have been generalized to the non-commutative case by Dvurečenskij.

We generalize to the non-commutative case a construction by Laskowski and Shashoua, which use an equivalence relation to obtain the Hájek decomposition.
Using equivalence classes, we obtain not only the Hájek decomposition, but also the Agliano-Montagna one, and the Cignoli-Esteva-Godo-Torrens one, thus establishing also the comparability of the three decompositions.